CYCLE-SUPERMAGIC COVERINGS AND DECOMPOSITION OF SOME GRAPHS

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ABSTRACT: A simple graph $G = (V, E)$ admits an $H$-covering if every edge in $E$ belongs to a subgraph of $G$ isomorphic to $H$. Further if all the subgraphs in the covering are edge-disjoint then the covering is said to be an $H$-decomposition of $G$. The graph $G$ is said to be $H$-magic if there is a total labeling $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, we have $\sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$ is constant and in this case $f$ is called a $H$-magic labeling. When $f(V) = \{1, 2, |V|\}$, then $G$ is said to be $H$-supermagic. An $H$-decomposition of $G$ is said to be an $H$-(super)magic decomposition of $G$ if $G$ has an $H$-(super)magic labeling. In this paper, we prove that the square graphs of bistar, path, cycle and the middle graph of cycle are $C_3$-supermagic. Further we show that the shadow graph of bistar admits $C_4$-supermagic decomposition and the middle graph of cycle admits $C_3$-supermagic decomposition.

Keywords: Total labeling, $H$-covering, $H$-supermagic covering, $H$-decomposition, $H$-supermagic decomposition. AMS Subject Classification(2010): 05C78.

1. INTRODUCTION

The concept of $H$-magic graphs was introduced by A. Gutiérrez and A. Lladó in [2]. A family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of different subgraphs of a graph $G$ is an edge-covering of $G$ if each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $1 \leq i \leq k$. Then, it is said that $G$ admits an $(H_1, H_2, \ldots, H_k)$-edge covering. If every $H_i$ is isomorphic to a given graph $H$, then we say that $G$ admits an $H$-covering.

If all subgraphs in the covering are edge-disjoint, the covering is also called an $H$-decomposition of $G$.

Suppose that $G = (V, E)$ admits an $H$-covering (or $H$-decomposition). We say that a bijective function $f : V \cup E \rightarrow \{1, 2, 3, \ldots, |V| + |E|\}$ is an $H$-magic labeling of $G$ if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of $G$ isomorphic to $H$, we have $f(H') = \sum_{v \in V'} f(v) + \sum_{e \in E'} f(e) = m(f)$. If $f(V) = \{1, 2, \ldots, |V|\}$, we say that $f$ is a $H$-supermagic labeling. An $H$-covering (or decomposition) of $G$ is said to be an $H$-(super)magic covering (or decomposition) of $G$ if $G$ has an $H$-(super)magic labeling. The constant value that every copy of $H$ takes under $f$ in the case of $H$-supermagic labeling is denoted by $s(f)$.

In [2], A. Gutiérrez and A. Lladó studied the families of complete and complete bipartite graphs with respect to the star-magic and star-supermagic properties and proved that the star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$, the complete graph $K_n$ is not $K_{1,h}$-magic for any $1 < h < n - 1$, the complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$ but is $K_{1,n}$-magic for $n \geq 1$. The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-supermagic for any integer $n > 1$ and for any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$. They also proved that the path $P_n$ is $P_{h}$-supermagic for any integer $2 \leq h \leq n$, the cycle $C_n$ is $P_{h}$-supermagic for some $h$. They also provided constructions of infinite families of $H$-magic graphs for an arbitrary given graph $H$. 
A. Lladó and Moragas [8] proved that the wheel $W_n$ is $C_3$-supermagic for $n$ is odd, the windmill $W(r; k)$ is $C_r$-supermagic, subdivided wheel $W_n(r; k)$ is $C_{2r+4}$-magic and $\theta_n(p)$ is $C_{2p}$-supermagic. In [9] T. K. Maryati et al. investigated $P_h$-supermagic labelings of some classes of trees such as the subdivision of stars, shrubs, and banana tree. $C_3$-supermagic labelings of generalized antiprism, triangular ladder, fan and $C_4$-supermagic labelings of prism and ladder graph can be found in [10], [11], [5] and [6]. Jeyanthi and Selvagopal proved that for an arbitrary 2-connected simple graph $H$ the chain graph [3], one point union of $n$ copies of $H$, the graph linear garland $LG_n(H)$ [7] and edge amalgamation of a finite number of graphs isomorphic to $H$ [4] are $H$-supermagic. In [4] they have also constructed two families of star-supermagic graphs. Same results were also proved by several authors in different methods.

We use the following notations. For any two integers $m < n$, we denote by $[m, n]$, the set of all consecutive integers from $m$ to $n$. For any set $\mathbb{I} \subseteq \mathbb{N}$ we write, $\Sigma \mathbb{I} = \sum_{x \in \mathbb{I}} x$ and for any integers $k, k + \mathbb{I} = \{k + x : x \in \mathbb{I}\}$. Thus $k + [m, n]$ is the set of consecutive integers from $k + m$ to $k + n$. It can be easily verified that $\Sigma(k + \mathbb{I}) = k|\mathbb{I}| + 1$. If $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ is a partition of a set $X$ of integers with the same cardinality then we say $\mathbb{P}$ is an $k$-equipartition of $X$. If $f$ is a total labeling on $G = (V, E)$ we denote $f(V) = \sum_{v \in V} f(v), f(E) = \sum_{e \in E} f(e)$ and $f(G) = \Sigma f(V) + \Sigma f(E)$.

**Definition 1.1.** [1] The bistar $B_{m,n}$ is the graph obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge.

**Definition 1.2.** [1] The shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, $G'$ and $G''$ and joining each vertex $u'$ in $G''$ to the neighbours of the corresponding vertex $v'$ in $G'$.

**Definition 1.3.** [1] For a simple connected graph $G$ the square of the graph $G$ is denoted by $G^2$ and is defined as the graph with the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance 1 or 2 apart in $G$.

**Definition 1.4.** [1] The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if either if they are adjacent edges of $G$ or one is a vertex of $G$ and other is an edge incident with it.

**Lemma 1.5.** [4] If $h$ is even, then there exists a $k$-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, hk]$ such that $\Sigma X_r = \frac{h(hk + 1)}{2}$ for $1 \leq r \leq k$. Thus, the subsets sum are equal and is equal to $\frac{h(hk + 1)}{2}$.

### 2. $C_3$-Supermagic Covering

In this section we prove that the square graphs of Bistars, paths and cycles are $C_3$-supermagic. Also we prove that the middle graph of cycles are also $C_3$-supermagic.

**Theorem 2.1.** The graph $B_{m,n}^2$ is $C_3$-supermagic for all $m, n \geq 1$.

**Proof.** Let $B_{m,n}$ be the bistar obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ with an edge. Let $V$ be the vertex set and $E$ be the edge set of the bistar $B_{m,n}^2$. Then, $V = \{u, v, u_i, v_i : 1 \leq i \leq m, 1 \leq j \leq n\}$ where $u$ and $v$ are the centers of $K_{1,m}$ and $K_{1,n}$ respectively and $u_i$ and $v_i$ are the pendant vertices of $K_{1,m}$ and $K_{1,n}$ respectively and $E = \{uv, uu_i, vuv_i, vv_i : 1 \leq i \leq m, 1 \leq j \leq n\}$. We have $|V| = m + n + 2$ and $|E| = 2(m + n) + 1$. 


For our convenience we rename the pendant vertices of the stars as follows: \( w_i = u_i \), for \( 1 \leq i \leq m \) and \( w_{m+i} = v_i \), for \( 1 \leq i \leq n \). Therefore, \( V = \{ w_i : 1 \leq i \leq m + n \} \). Let \( H_i \) be the 3-cycle uwivu for \( 1 \leq i \leq m + n \).

Clearly \( \{ H_i : 1 \leq i \leq m + n \} \) is a \( C_3 \)-covering for \( B_{m,n}^2 \). We prove the theorem in two cases.

**Case (i):** \( m \) and \( n \) are of different parity.

Define a total labeling \( f : V \cup E \to \{1, 2, \ldots, m+n\} \) as follows.

\[
f(w_i) = m + n + 1 - i \quad \text{for} \quad 1 \leq i \leq m + n;
\]

\[
f(u) = m + n + 1 \quad \text{and} \quad f(v) = m + n + 2.
\]

For \( 1 \leq i \leq \frac{m+n-1}{2} \),

\[
f(H_i) = f(u) + f(w_i) + f(v) + f(uw_i) + f(vw_i) + f(uv)
= m + n + 1 + m + n + 2 + m + n + 1 - i + \frac{3(m + n) + 5}{2} - i
+ 2(m + n) + 2 + 2i + 3(m + n + 1)
= 9(m + n + 1) + \frac{m+n+5}{2}.
\]

For \( \frac{m+n-1}{2} \leq i \leq m + n \),

\[
f(H_i) = f(u) + f(w_i) + f(v) + f(uw_i) + f(vw_i) + f(uv)
= m + n + 1 + m + n + 2 + m + n + 1 - i + \frac{5(m + n) + 5}{2} - i
+ m + n + 2 + 2i + 3(m + n + 1)
= 9(m + n + 1) + \frac{m+n+5}{2}.
\]
Thus, \( f(H_i) = 9(m + n + 1) + \frac{m + n + 5}{2} \) which is a constant for \( 1 \leq i \leq m + n \).

Hence, \( \{H_i: 1 \leq i \leq m + n \} \) is a \( C_3 \)-supermagic covering for \( B_{m,n}^2 \) and \( f \) is a \( C_3 \)-supermagic labeling with supermagic sum \( 9(m + n + 1) + \frac{m + n + 5}{2} \).

**Case (ii):** \( m \) and \( n \) are of same parity.

Define a total labeling \( f: V \cup E \to \{1, 2, \cdots, m + n\} \) as follows.

\[
\begin{align*}
  f(w_i) &= m + n + 1 - i \quad \text{for} \quad 1 \leq i \leq m + n; \\
  f(u) &= m + n + 1 \quad \text{and} \quad f(v) = m + n + 2.
\end{align*}
\]

\[
\begin{align*}
  f(uw_i) &= \begin{cases} 
    \frac{3(m+n+2)}{2} - i & \text{for} \quad 1 \leq i \leq \frac{m+n-1}{2} \\
    \frac{5(m+n)+8}{2} - i & \text{for} \quad \frac{m+n+1}{2} \leq i \leq m + n
  \end{cases} \\
  f(vw_i) &= \begin{cases} 
    \frac{2(m+n)+3+2i}{2} & \text{for} \quad 1 \leq i \leq \frac{m+n-1}{2} \\
    m + n + 2 + 2i & \text{for} \quad \frac{m+n+1}{2} \leq i \leq m + n
  \end{cases}
\end{align*}
\]

and \( f(uv) = \frac{3(m+n)}{2} + 3 \).

We can easily prove that \( f(H_i) = 8(m + n) + 13 \) which is a constant for \( 1 \leq i \leq m + n \). Hence, \( \{H_i: 1 \leq i \leq m + n \} \) is a \( C_3 \)-supermagic covering for \( B_{m,n}^2 \) and \( f \) is a \( C_3 \)-supermagic labeling with supermagic sum \( 8(m + n) + 13 \).

From both the cases we have \( B_{m,n}^3 \) is \( C_3 \)-supermagic for all \( m, n \geq 1 \).

**Illustration 2.2.** \( C_3 \)-supermagic labelings of \( B_{4,3}^2 \) and \( B_{4,4}^2 \) are given in Figure 1(a): C3-supermagic labelings of \( B_{4,3}^2 \) with supermagic sum 78.
Theorem 2.3. $P_n^2$ is $C_3$-supermagic for $n \geq 4$.

Proof. Let $P_n = v_1 v_2 e_2 \cdots v_{n-1} e_{n-1} v_n$ be a path. Then $e_i = v_i v_{i+1}$ for $1 \leq i \leq n-1$. Let $P_n^2$ be the square graph of $P_n$. Let $V$ be the vertex set and $E$ be the edge set of $P_n^2$. Then, $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_i : 1 \leq i \leq n-1\} \cup \{e'_j : 1 \leq j \leq n-2\}$ where $e_i = v_i v_{i+1}$ and $e'_j = v_j v_{j+2}$. $|V| = n$ and $|E| = 2n-3$.

Define a total labeling $f : V \cup E \rightarrow \{1, 2, \ldots, 3n-3\}$ as follows.

$f(v_i) = i$ for $1 \leq i \leq n$,

$f(e_i) = 2n-i$ for $1 \leq i \leq n-1$,

$f(e'_i) = 3n-2-i$ for $1 \leq i \leq n-2$.

Let $H_i$ be the 3-cycle $v_i v_{i+1} v_{i+2} e'_i v_i$ for $1 \leq i \leq n-2$. Then $\{H_1, H_2, \ldots, H_{n-2}\}$ is a $C_3$-covering for $P_n^2$.

\[
f(H_i) = f(v_i) + f(e_i) + f(v_{i+1}) + f(e_{i+2}) + f(e'_i) = i + 2n - i + i + 1 + 2n - i - 1 + i + 2 + 3n - 2 - i = 7n.
\]

Hence, $f(H_i) = 7n$ which is a constant for $1 \leq i \leq n-2$. Further, $f(V) = \{1, 2, \ldots, |V| + |E|\}$. Hence, $\{H_1, H_2, \ldots, H_{n-2}\}$ is a $C_3$-supermagic covering and $f$ is a $C_3$-supermagic labeling with supermagic sum $7n$. Therefore, the square graph $P_n^2$ is $C_3$-supermagic for $n \geq 4$.

Illustration 2.4. $C_3$-supermagic labelings of $P_n^2$ is given in Figure 2.

Theorem 2.5. $C_n^2$ is $C_3$-supermagic for $n \geq 7$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$ where $n \geq 7$. Let $V$ be the vertex set and $E$ be the edge set of $C_n^2$. Then $V = \{v_i : 1 \leq i \leq n\}$ and $E = \{v_i v_{i+1}, v_i v_{i+2} : 1 \leq i \leq n\}$ where the for $i$ is taken modulo $n$. Note that $|V| = n$ and $|E| = 2n$. 

Figure 1(b): $C_3$-supermagic labelings of $B_4^2$ with supermagic sum 77.
Define a total labeling $f : V \cup E \rightarrow \{1, 2, \cdots, 3n\}$ as follows:

$$f(v_i) = i \text{ for } 1 \leq i \leq n,$$

$$f(v_iv_{i+1}) = 2n - i \text{ for } 1 \leq i \leq n - 1,$$

$$f(v_nv_1) = 2n,$$

$$f(v_iv_{i+2}) = 3n - i + 1 \text{ for } 1 \leq i \leq n, \text{ } i \text{ is taken modulo } n.$$

Let $H_i$ be the 3-cycle $v_iv_{i+1}v_{i+2}$ for $1 \leq i \leq n$ and the subscript $i$ is taken modulo $n$. Then $\{H_1, H_2, \cdots, H_n\}$ is a $C_3$-covering for $C_n^2$.

$$f(H_i) = f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(v_{i+1}v_{i+2}) + f(v_{i+2}) + f(v_{i+1}v_1) = i + 2n - i + i + 1 + 2n - i - 1 + i + 2 + 3n - i + 1 = 7n + 3.$$  

Hence, $f(H_i) = 7n + 3$ which is a constant for $1 \leq i \leq n$. Further, $f(V) = \{1, 2, \cdots, |V| + |E|\}$. Hence, $\{H_1, H_2, \cdots, H_n\}$ is a $C_3$-supermagic covering and $f$ is a $C_3$-supermagic labeling with supermagic sum $14n + 3$.

Therefore, the square graph $C_n^2$ is $C_3$-supermagic.

Figure 2: $C_3$-supermagic labelings of $P_n^2 \times P_n$ with supermagic sum 49.

Figure 3: $C_3$-supermagic labelings of $C_n^2$ with supermagic sum 59.
Theorem 2.6. If \( C_n \) is a cycle graph then the middle graph of the cycle graph \( M(C_n) \) is \( C_3 \)-supermagic.

Proof. Let \( v_1, v_2, \cdots, v_n \) be the vertices of the cycle \( C_n \) and \( v'_1, v'_2, \cdots, v'_n \) be the newly inserted vertices corresponding to the \( n \) edges \( e_1, e_2, \cdots, e_n \) of \( C_n \) to obtain \( M(C_n) \).

Let \( V \) be the vertex set and \( E \) be the edge set of \( M(C_n) \). Then \( V = \{ v_i, v'_i : 1 \leq i \leq n \} \) and \( E = \{ e_i : 1 \leq i \leq n \} \cup \{ v'_i v_{i+1}, v'_i v'_i : 1 \leq i \leq n - 1 \} \cup \{ v'_n v'_1 : 1 \leq i \leq n - 1 \} \). Note that \( |V| = 2n \) and \( |E| = 3n \).

Define a total labeling \( f : V \cup E \to \{ 1, 2, \cdots, 5n \} \) as follows:

\[
\begin{align*}
  f(v_i) &= i \text{ for } 1 \leq i \leq n, \\
  f(v'_i) &= 2n + 1 - i \text{ for } 1 \leq i \leq n, \\
  f(v_i v'_i) &= 2n + i \text{ for } 1 \leq i \leq n, \\
  f(v'_i v'_{i+1}) &= 4n - i \text{ for } 1 \leq i \leq n - 1, \\
  f(v'_n v'_1) &= 4n, \\
  f(v'_i v'_i) &= 4n + i \text{ for } 1 \leq i \leq n, \\
  f(v'_n v'_1) &= 5n.
\end{align*}
\]

Let \( H_i \) be the 3-cycle \( v'_i v'_{i+1} v_{i+1} \) for \( 1 \leq i \leq n \) and the subscript \( i \) is taken modulo \( n \). Then \( \{ H_1, H_2, \cdots, H_n \} \) is a \( C_3 \)-covering for \( M(C_n) \).

\[
f(H_i) = f(v'_i) + f(v'_{i+1}) + f(v_{i+1}) + f(v'_i v_{i+1}) + f(v'_i v'_{i+1}) + f(v'_n v'_1) = 2n + 1 - i + 4n - i + i + 1 + 2n + i + 1 + 2n - i + 4n + i = 14n + 3.
\]

Hence, \( f(H_i) = 14n + 3 \) which is a constant for \( 1 \leq i \leq n \). Further, \( f(V) = \{ 1, 2, \cdots, |V| + |E| \} \). Hence, \( \{ H_1, H_2, \cdots, H_n \} \) is a \( C_3 \)-supermagic covering and \( f \) is a \( C_3 \)-supermagic labeling with supermagic sum \( 14n + 3 \). Therefore, the middle graph of the cycle graph \( M(C_n) \) is \( C_3 \)-supermagic.

Illustration 2.7. \( C_3 \)-supermagic labelings of \( M(C_8) \) is given in Figure 3.

![Figure 4: \( C_3 \)-supermagic labelings of \( M(C_8) \) with supermagic sum 115.](image-url)
3. CYCLE-SUPERMAGIC DECOMPOSITION

Theorem 3.1. The shadow graph $D_2(B_{m,n})$ of the bistar $B_{m,n}$ admits $C_4$-supermagic decomposition.

Proof. Consider two copies of $B_{m,n}$. Let \{u', v', u'', v'' : 1 \leq i \leq m \} and \{u''', v''' : 1 \leq i \leq n \}$ be the corresponding vertex sets of each copy of $B_{m,n}$. Let $V$ be the vertex set and $E$ be the edge set of $D_2(B_{m,n})$. Then $|V| = 2(m + n + 2)$ and $|E| = 4(m + n + 1)$.

Let $H_i'$ be the 4-cycle $u_i'u_i''u_i''u_i'$ for $1 \leq i \leq m$, $H_i''$ be the 4-cycle $v_i'v_i''v_i''v_i'$ for $1 \leq i \leq n$ and $H = u'v'u''v'$. Then \{$H_i', H_i'', H_i : 1 \leq i \leq m, 1 \leq j \leq n \}$ is a $C_4$-decomposition for $D_2(B_{m,n})$.

By Lemma 1.5, \[1, 4(m + n + 1)] \] can be partitioned into \[(m + n + 1)\]-equipartitions $\mathbb{P} = \{X_1', X_2', \ldots, X_{m+n+1}'\}$ each containing four integers such that $\Sigma X_i' = 8(m + n) + 10$ for $1 \leq i \leq m + n + 1$. Add $2(m + n + 2)$ to $[1, 4(m + n + 1)]$. Then $[2(m + n + 2) + 1, 6(m + n) + 8]$ is partitioned into $m + n + 1$-equipartitions $\mathbb{P} = \{X_1, X_2, \ldots, X_{m+n+1}\}$ such that $\Sigma X_i = 8(m + n + 2) + 8(m + n) + 10 = 16(m + n) + 26$.

Define a total labeling $f : V \cup E \rightarrow \{1, 2, \ldots, 6(m + n) + 8\}$ as follows:

\[
\begin{align*}
  f(u_i') &= i + 2 & \text{for } 1 \leq i \leq m, \\
  f(u_i'') &= 2m + 2n + 3 - i & \text{for } 1 \leq i \leq m, \\
  f(v_i') &= m + i + 2 & \text{for } 1 \leq i \leq n, \\
  f(v_i'') &= m + 2n + 3 - i & \text{for } 1 \leq i \leq n, \\
  f(u'') &= 1, \\
  f(u''') &= 2(m + n) + 4, \\
  f(v') &= 2, \\
  f(v''') &= 2(m + n) + 3 \\
  f(E(H_i')) &= X_i & \text{for } 1 \leq i \leq m, \\
  f(E(H_i'')) &= X_i & \text{for } m + 1 \leq i \leq m + n, \\
  f(E(H)) &= X_{m+n+1}.
\end{align*}
\]

Then we have

\[
\begin{align*}
  f(H_i) &= f(u_i') + f(u_i'') + f(u''') + f(E(H_i')) \\
  &= 4(m + n) + 10 + 16(m + n) + 26 = 20(m + n) + 36 \text{ for } 1 \leq i \leq m, \\
  f(H_i'') &= f(v_i') + f(v_i'') + f(v''') + f(E(H_i'')) \\
  &= 20(m + n) + 36 \text{ for } 1 \leq i \leq n, \\
  f(H) &= f(u') + f(v') + f(u'') + f(v''') + f(E(H)) \\
  &= 20(m + n) + 36.
\end{align*}
\]

Hence, \{$H_i', H_i'', H_i : 1 \leq i \leq m, 1 \leq j \leq n \}$ is a $C_4$-decomposition for $D_2(B_{m,n})$ with supermagic sum $s(f) = 20(m + n) + 36$.

Illustration 3.2. $C_4$-supermagic decomposition of $D_2(B_{3,2})$ is given in Figure 5.
Theorem 3.3. The middle graph $M(C_n)$ of the cycle $C_n$ admits $C_3$-supermagic decomposition.

Proof. The $C_3$-covering $\{H_1, H_2, \ldots, H_n\}$ we defined in Theorem 2.6 itself is a $C_3$-supermagic decomposition for $M(C_n)$. Hence, $M(C_n)$ admits a $C_3$-supermagic decomposition.

References


