ON G-FINITISTIC L-TOPOLOGICAL SPACES

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ABSTRACT: In this paper we have generalized the concept of Finitisticness of L-topological Spaces via grills, introduced the concept of G- Finitistic L-topological space and studied its various basic properties.

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1. INTRODUCTION AND PRELIMINARIES

The concept of finitistic space in general topology was introduced in 1960 by R. G. Swan\(^1\). Of course Swan did not name these spaces as finitistic. The term “Finitistic” was used by Bredon in 1972 in his book\(^2\) and since then it has become a firmly established term. In general topology the study of finitistic spaces became interesting with the basic paper of Deo-Tripathi\(^3\) in 1982 in which they proved a Characterization Theorem for non-finitistic paracompact spaces. Several works appear there after [32], one can see (8, 29, 30, 31, 33, 34, 35, 36 and 38). In these papers almost all the possible properties of finitistic spaces in general topology has been studied and concepts like countably finitistic, locally finitistic and completely finitistic spaces have been introduced and their properties were almost settled completely. Jamwal-Shakeel\(^4\) introduced the concept of Finitistic Space in Fuzzy and L-Fuzzy topology in 2002 and proved several properties. Our further work in this direction can be seen in references\(^5\) to \(^29\).

In the year 1965, An American Mathematician L. A. Zadeh\(^6\) introduced the concept of Fuzzy Set and studied its various properties. Thus a new type of set theory took birth, which is known as fuzzy set theory. This historic paper of Zadeh enthralled mathematicians all over the world and they started to study almost all the mathematical concepts based on Cantors set theory in terms of fuzzy set theory. In this way a new branch of mathematics started to emerge which is today known as fuzzy mathematics. In 1968, a Chinese mathematician, C. L. Chang\(^7\) introduced the concept of fuzzy topology and developed basic notions for these spaces. After the paper [3] of Chang, many papers throughout the World were published on this subject. In 1967 and 1973 Goguen\(^8\) introduced the concepts of fuzzy subset and fuzzy topology respectively and introduced the concepts of L-fuzzy subset and L-fuzzy topology. The first book\(^9\) on fuzzy topology by Chinese mathematicians Liu-Luo was published in 1997. Hohle-Rodabaugh\(^10\) used the terms L-topology and L-topological space for the terms L-fuzzy topology and L-fuzzy topological Space. In our work we have also used terms L-topology and L-topological space for the terms L-fuzzy topology and L-fuzzy topological Space.

Let \((X,\delta)\) be an L-topological space. A subfamily \(\mu = \{U_\lambda: \lambda \in \Delta\}\) of \(\delta\) is said to be an open cover\(^11\) of \((X, \delta)\) if \(\forall l \in D\), \(U_1 = 1\). A subfamily \(n = \{V_\alpha: \alpha \in \Delta\}\) of \(\delta\) is said to be an open refinement of \(\mu\) if \(n = \{V_\alpha: \alpha \in \Delta\}\) is an open cover of \((X, \delta)\) and for each \(V_\alpha \in n\) there exists some \(U_\lambda \in m\) such that \(V_\alpha \leq U_\lambda\). Let \(A\) be an L-fuzzy subset of \(X\). A subfamily \(\{U_\lambda : l \in \Lambda\}\) of \(\delta\) is said to be an open cover of \(A\) in an L-topological space \((X, \delta)\) if \(\forall \lambda \in \Lambda\), \(U_\lambda \geq A\).
Let $(X, \delta_1)$ and $(Y, \delta_2)$ be two L-topological spaces. A function $f: X \to Y$ is said to be \textbf{continuous} from $(X, \delta_1)$ to $(Y, \delta_2)$ if $\forall \ U \in \delta_2 \Rightarrow Uf \in \delta_1$. $f$ is said to be \textbf{homeomorphism} from $(X, \delta_1)$ to $(Y, \delta_2)$ if both and $f^{-1}$ are continuous from $(X, \delta_1)$ to $(Y, \delta_2)$ and from $(Y, \delta_2)$ to $(X, \delta_1)$ respectively and $f$ is also bijection (Pages 437 to 439 of reference no.[39]). (Note $Uf$ means $U \circ f$).

Let $G$ be a Group and $(L, ')$ be a complete lattice with atleast two elements. Then it can be easily shown that $\delta = \{U \in L^G: U(x) = U(x^{-1}), \forall x \in G\}$ is an L-topology on G. This L-topology is called \textbf{gL-topology} and $(G, \delta)$ is called gL-topological space. Let $G$ be a Group and $(L, ')$ be a complete lattice with atleast two elements. Then it can be easily shown that $\delta = \{U \in L^G: U(x) = U(x^n), \forall x \in G$ and $\forall n \in Z\}$ is an L-topology on G. This L-topology is called \textbf{gil-topology} and $(G, \delta)$ is called gil-topological space.

Let $X$ be a nonempty set. A \textbf{Grill} on $X$ is a nonempty collection $G$ of nonempty subsets of $X$ satisfying two conditions (i) $U \in G$ and $U \subseteq V \Rightarrow V \in G$ (ii) $U \cap V \in G \Rightarrow U \vee V \in G$. Let $(X, T)$ be a topological space and $G$ be a Grill on $X$. The concept of Grill can be easily generalized for L-fuzzy Subsets as: A collection $G$ of L-fuzzy subsets of $X$ satisfying three conditions (i) $0 \notin G$ (ii) $U \in G$ and $U \subseteq V \Rightarrow V \in G$ (ii) $U \cup V \in G \Rightarrow U \vee G$ or $V \in G$ is an L-fuzzy Grill on $X$.

Let $X$ be a nonempty set. A \textbf{fuzzy Ideal} on $X$ is a nonempty collection $I$ of fuzzy subsets of $X$ such that (1) $U \in I$ and $V \subseteq U \Rightarrow V \in I$ (ii) $U, V \in I \Rightarrow U \cup V \in I$. Let $X$ be a nonempty set. An L-fuzzy Ideal on $X$ is a nonempty collection $I$ of L-fuzzy subsets of $X$ such that (1) $U \in I$ and $V \subseteq U \Rightarrow V \in I$ (ii) $U, V \in I \Rightarrow U \vee V \in I$.

Let $\Delta \neq \emptyset$ and $A = \{A_{\lambda}, \lambda \in \Delta\}$ be a family of L-fuzzy subsets of a nonempty set $X$. Then the \textbf{order} $(a, b, c)$ of $A$ is defined as under:

\textbf{Case-I.} When $A_{\lambda} \neq 0$ for atleast one value of $\lambda$ in $\Delta$. Then the order of $A$ is the largest nonnegative integer $n$ for which there exists a subset $M$ of $\Delta$ having $n + 1$ elements such that $\bigwedge_{\lambda \in M} A_{\lambda} \neq 0$ or is $\infty$ if there is no such largest integer $n$.

\textbf{Case-II.} When $A_{\lambda} = 0$ for all $\lambda \in \Delta$. Then the order of $A$ is $-1$.

An L-topological space $(X, \delta)$ is said to be \textbf{finitistic} \cite{5,6,7} if every open cover of $(X, \delta)$ has a finite order open refinement. An L-topological space $(X, \delta)$ is said to be \textbf{L-finitistic} \cite{27} if every open cover $\mu$ of $(X, \delta)$ there exists a finite order subfamily $\nu = \{V_\alpha: \alpha \in \Delta\}$ of $\delta$ such that $(\bigvee_{\alpha \in \Delta} V_\alpha) \in I$ and for each $V_\alpha \in \nu$ there exists some $U_\lambda \in \mu$ such that $V_\alpha \subseteq U_\lambda$.

All the other undefined terms on fuzzy topology/L-fuzzy topology which have been used in this paper can easily be seen in the reference no. \cite{40}.

\section{2. G- Finitistic L-topological Spaces}

\textbf{Definition 2.1.} Let $(X, \delta)$ be an L-topological Space and $G$ be an L-fuzzy Grill on $X$. Then an L-topological space $(X, \delta)$ is said to be G-Finitistic if every open cover $\mu$ of $(X, \delta)$ there exists a subfamily $\nu = \{V_\alpha: \alpha \in \Delta\}$ of $\delta$ of order not exceeding $n$ such that $(\bigvee_{\alpha \in \Delta} V_\alpha)' \notin G$ and for each $V_\alpha \in \nu$ there exists some $U_\lambda \in \mu$ such that $V_\alpha \subseteq U_\lambda$.

\textbf{Theorem 2.2.} Every finitistic L-topological Space is G-Finitistic.

\textbf{Proof.} Let $(X, \delta)$ be a finitistic L-topological space and $G$ is an L-fuzzy Grill on $X$. We have to show that $(X, \delta)$ is G-Finitistic. Let $\mu$ be any open cover of $(X, \delta)$. Since $(X, \delta)$ is finitistic, therefore $\mu$ has a finite order open refinement say $\nu = \{V_\alpha: \alpha \in \Delta\}$. Since $\nu = \{V_\alpha: \alpha \in \Delta\}$ is an open cover of $(X, \delta)$, therefore $(\bigvee_{\alpha \in \Delta} V_\alpha)' \notin G$ and for each $V_\alpha \in \nu$ there exists some $U_\lambda \in \mu$ such that $V_\alpha \subseteq U_\lambda$. 


exists a finite order subfamily \( H \). Hence by Theorem 2.5 (X, \( \alpha \)) is \( \alpha \)-Finitistic. It means (X, \( \alpha \)) is \( \alpha \)-Finitistic.

In the following result one can see a relationship between Finitisticness of a general topological space and G-Finitisticness of an L-Topological Space.

Theorem 2.7. Let \((X, T)\) be a topological space and L is any complete lattice. Then \((X, T)\) is finitistic if and only if \((X, \text{crs}(\delta))\) is G-finitistic where \( G = L^X - \{0\} \) and \( \delta = \{\chi_U : U \subseteq T\} \).

Proof. Clearly \( G = L^X - \{0\} \) is an L-fuzzy Grill on X and \( \delta = \{\chi_U : U \subseteq T\} \) is an L-topology on X. Let \((X, \delta)\) be a G-finitistic L-topological space. Let \( \delta \) be any open cover of \((X, T)\). Then it can be easily checked that \( \mu_1 = \{\chi_U : U \subseteq \mu\} \) is an open cover of \((X, \delta)\). Since \((X, \delta)\) is G-finitistic, there exists a finite order subfamily \( v = \{V_{\alpha} : \alpha \in \Delta\} \) of \( \delta \) such that \( (\bigvee_{a \in \Delta} V_{\alpha})' \notin G \) and for each \( V_{\alpha} \in v \) there exists some \( U_{\alpha} \subseteq \mu \) such that \( V_{\alpha} \leq U_{\alpha} \). But \( (\bigvee_{a \in \Delta} V_{\alpha})' \notin G \Rightarrow (\bigvee_{a \in \Delta} \chi_{V_{\alpha}}) = 1 \Rightarrow \chi_{V_{\alpha}} \subseteq \Delta \) is a finite order open refinement of \( \mu_1 \). It can be easily checked that \( v = \{V_{\alpha} : \alpha \in \Delta\} \) is a finite order open refinement of \( \delta \). It means (X, \( \delta \)) is G-Finitistic.

Theorem 2.8. Let \((X, \delta)\) be an L-topological space. Then \((X, [\delta])\) is finitistic if and only if \((X, \text{crs}(\delta))\) is G-finitistic where \( G = L^X - \{0\} \).

Proof follows by Theorem 2.7.

Remark 2.9. If \((X, \delta)\) is an L-topological space where \( \delta \) is finite, then \((X, \delta)\) is G-finitistic.

Remark 2.10. In general topology we know \((X, T)\) is finitistic when X is finite. But this result needs not be true in L-topology because we know in L-topology \((X, \delta)\) needs not be finitistic when X is finite. Hence by Theorem 2.5 (X, \( \delta \)) needs not be G-finitistic when X is finite.

Theorem 2.11. If \((X, \delta)\) is an L-topological space and G, J are two L-fuzzy Grills on X such that \( G \subseteq J \), then \((X, \delta)\) is J-finitistic \( \Rightarrow \) (X, \( \delta \)) is G-finitistic.

Proof. Let \((X, \delta)\) be J-Finitistic. Let \( \mu \) be any open cover of \((X, \delta)\). Since \((X, \delta)\) is J-Finitistic, there exists a finite order subfamily \( v = \{V_{\alpha} : \alpha \in \Delta\} \) of \( \delta \) such that \( (\bigvee_{a \in \Delta} V_{\alpha})' \notin J \) and for each \( V_{\alpha} \in v \) there exists some \( U_{\alpha} \subseteq \mu \) such that \( V_{\alpha} \leq U_{\alpha} \). But \( (\bigvee_{a \in \Delta} V_{\alpha})' \notin J \Rightarrow (\bigvee_{a \in \Delta} \chi_{V_{\alpha}}) \notin G \Rightarrow (X, \delta) \) is G-finitistic.

Theorem 2.12. Let \((X, \delta_1)\) and \((Y, \delta_2)\) be two L-topological spaces and \( f : (X, \delta_1) \rightarrow (Y, \delta_2) \) be a homeomorphism. Then \((X, \delta_1)\) is G-finitistic if and only if \((Y, \delta_2)\) is \( f(G) \)-finitistic.

Proof. Here \((X, \delta_1)\) and \((X, \delta_1)\) are two L-topological spaces and \( f : (X, \delta_1) \rightarrow (Y, \delta_2) \) is a homeomorphism. Here G is an L-fuzzy Grill on X. It can be easily shown that \( f(G) = \{\{A\} : A \in G\} \) is an L-fuzzy Grill on Y. Let \((X, \delta_1)\) be G-finitistic. Let \( \mu = \{U_{\lambda} : \lambda \in \Delta\} \) be any open cover of \((Y, \delta_2)\). Then it can be easily checked that \( \{f(U_{\lambda} : U_{\lambda} \in \mu)\} \) is an open cover of \((X, \delta_1)\). Since \((X, \delta_1)\) is G-finitistic, there exists a finite order open refinement of \( \mu \). Hence by Theorem 2.5 (X, \( \delta \)) is G-Finitistic.
A subfamily say \( v = \{ V_\alpha : \alpha \in \Delta \} \) of \( \delta_1 \) such that \((\forall_{\alpha \in \Delta} V_\alpha)' \not\in G \) and for each \( V_\alpha \in v \) there exists some \( U_\lambda \in \mu \) such that \( V_\alpha \cap U_\lambda = \emptyset \). Let \((\forall_{\alpha \in \Delta} V_\alpha)' = A \not\in G \). Then \( A F^{-1} \not\in f(G) \). Since \( f : (X, \delta_1) \to (Y, \delta_2) \) is a homeomorphism, therefore it can be easily seen that \( V_1 = \{ V_\alpha f^{-1} : V_\alpha \in v \} \) is a finite order subfamily of \( \delta_2 \) such that \((\forall_{\alpha \in \Delta} V_\alpha)' \not\in f(G) \) and for each \( V_\alpha f^{-1} \in v_1 \) there exists some \( U_\lambda \in \mu \) such that \( V_\alpha f^{-1} \subseteq U_\lambda \). Hence \((Y, \delta_2) = f(G)\)-finitistic. Similarly converse can be proved.

**Remark 2.14.** The above result is not true for continuous functions because this result is not true even for Finitisticness of L-topological spaces.

**Theorem 2.16.** Let \( G \) be an L-fuzzy grill on \( X \) such that \( G|_Y \) is an L-fuzzy grill on \( Y \). Then every closed subspace \((Y, \delta|_Y)\) of \( G\)-finitistic L-topological space \((X, \delta)\) is \( G|_Y \)-finitistic.

**Proof.** Here \((X, \delta)\) is \( G\)-finitistic L-topological and \((Y, \delta|_Y)\) is a closed subspace of \((X, \delta)\). We have to show that \((Y, \delta|_Y) = G|_Y\)-finitistic. Let \( \mu = \{ U_\lambda : \lambda \in \Delta \} \) be any open cover of \((Y, \delta|_Y)\). Then for each \( U_\lambda \in \mu \), there exists some \( V_\lambda \in \delta \) such that \( U_\lambda = V_\lambda|_Y \). Since \((Y, \delta|_Y)\) is a closed subspace of \((X, \delta)\), therefore \( \gamma|_Y \in \delta \). Clearly \( v = \{ V_\lambda : U_\lambda = V_\lambda|_Y \text{ and } U_\lambda \in \mu \} \cup \{ \gamma|_Y \} \) is an open cover of \((X, \delta)\). Since \((X, \delta)\) is \( G\)-finitistic, there exists a finite order subfamily \( v_1 = \{ W_\alpha : \alpha \in \Delta \} \) of \( \delta \) such that \((\forall_{\alpha \in \Delta} W_\alpha)' \not\in G \) and for each \( W_\alpha \in v \) there exists some \( S_\alpha \in v \) such that \( W_\alpha \subseteq S_\alpha \). Then clearly the \( \mu_1 = \{ W_\alpha|_Y : W_\alpha \in v_1 \} \) is a finite order subfamily of \( \delta \) such that \((\forall_{W_\alpha|_Y \in \mu_1} (W_\alpha|_Y)' \not\in G \) and for each \( W_\alpha|_Y \in \mu_1 \) there exists some \( V_\lambda \in \mu \) such that \( W_\alpha|_Y \subseteq V_\lambda \). Hence \((Y, \delta|_Y) = G|_Y\)-finitistic.

**Theorem 2.17.** Let \((X, \delta)\) be any L-topological space. Let \( I \) and \( G \) be respectively \( L\)-fuzzy Ideal and \( L\)-fuzzy Grill on \( X \) such that \( I \cup G = L^X \) and \( I \cap G = \emptyset \). Then \((X, \delta)\) is \( I\)-finitistic if and only if \((X, \delta)\) \( G\)-finitistic.

**Proof.** Let \((X, \delta)\) be \( I\)-finitistic. Let \( \mu \) be any open cover of \((X, \delta)\). Since \((X, \delta)\) is \( I\)-finitistic, there exists a finite order subfamily say \( v = \{ V_\alpha : \alpha \in \Delta \} \) of \( \delta \) such that \((\forall_{\alpha \in \Delta} V_\alpha)' \in I \). Also for each \( V_\alpha \in v \) there exists some \( U_\lambda \in \mu \) such that \( V_\alpha \subseteq U_\lambda \). Since \( I \cup G = L^X \) and \( I \cap G = \emptyset \), therefore \((\forall_{\alpha \in \Delta} V_\alpha)' \in I \Rightarrow (\forall_{\alpha \in \Delta} V_\alpha)' \not\in G \). It means \((X, \delta)\) is \( G\)-finitistic.

**Conversely.** Suppose \((X, \delta)\) is \( G\)-finitistic. Let \( \mu \) be any finite open cover of \((X, \delta)\). Since \((X, \delta)\) is \( G\)-finitistic, there exists a finite order subfamily say \( v = \{ V_\alpha : \alpha \in \Delta \} \) of \( \delta \) such that \((\forall_{\alpha \in \Delta} V_\alpha)' \not\in G \). Also for each \( V_\alpha \in v \) there exists some \( U_\lambda \in \mu \) such that \( V_\alpha \subseteq U_\lambda \). Since \( I \cup G = L^X \) and \( I \cap G = \emptyset \), therefore \((\forall_{\alpha \in \Delta} V_\alpha)' \not\in G \Rightarrow (\forall_{\alpha \in \Delta} V_\alpha)' \in I \). It means \((X, \delta)\) is \( I\)-finitistic.

**Remark 2.18.** Let \( \delta_1 \) and \( \delta_2 \) be two topologies on \( X \) such that \( \delta_1 \subseteq \delta_2 \). Then \((X, \delta_1)\) is \( G\)-Finitistic need not to imply that \((X, \delta_2)\) is \( G\)-finitistic and similarly \((X, \delta_2)\) is \( G\)-finitistic need not to imply that \((X, \delta_2)\) is \( G\)-finitistic.

**Example 2.20.** Let \((X, \delta)\) be a \( gL\)-topological Space where \( L = \{0, 1\} \). The \((X, \delta)\) is \( G\)-finitistic where \( G \) is any grill on \( X \).

## 3. 3. G-FINITISTIC L-FUZZY SUBSETS

**Definition 3.1.** Let \((X, \delta)\) be any L-topological Space and \( G \) be an L-fuzzy Grill on \( X \). Let \( A \) be an L-fuzzy subset of \( X \). An L-fuzzy subset \( A \) of \( X \) is said to be \( G\)-finitistic in \((X, \delta)\) if each open cover \( \mu \) of \( A \) there exists a finite order subfamily \( v = \{ V_\alpha : \alpha \in \Delta \} \) of \( \delta \) such that \((\forall_{\alpha \in \Delta} V_\alpha)' \not\in G \) and for each \( V_\alpha \in v \) there exists some \( U_\lambda \in \mu \) such that \( V_\alpha \subseteq U_\lambda \).

**Theorem 3.2.** Let \( A \) and \( B \) be two \( G\)-finitistic L-fuzzy subsets of an L-topological Space \((X, \delta)\). Then \( A \cup B \) is also \( G\)-finitistic in \((X, \delta)\).
**Proof.** Let \( \mu \) be any open cover of \( A \cup B \). Then clearly \( \mu \) is an open cover of \( A \) as well as \( B \). Since \( A \) and \( B \) are \( G \)-finitistic, there exists two finite order subfamilies say \( \mu_1 = \{ V_{\alpha} : \alpha \in \Delta_1 \} \) and \( \mu_2 = \{ W_{\beta} : \beta \in \Delta_2 \} \) of \( \delta \) such that \( (\bigcup_{\alpha \in \Delta_1} V_{\alpha})' \not\in G \) and \( (\bigcup_{\beta \in \Delta_2} W_{\beta})' \not\in G \) and for all \( V_{\alpha} \in \mu_1 \) and for all \( W_{\beta} \in \mu_2 \) there exists \( U_1 \) and \( U_2 \) in \( \mu \) such that \( V_{\alpha} \leq U_1 \) and \( W_{\beta} \leq U_2 \). Let \( \nu = \nu_1 \cup \nu_2 \). Then clearly \( \nu \) is a finite order subfamily of \( \delta \) such that \( (\bigcup_{\alpha \in \Delta_1} V_{\alpha})' \vee (\bigcup_{\beta \in \Delta_2} W_{\beta})' = (\bigcup_{\alpha \in \Delta_1} V_{\alpha})' \wedge (\bigcup_{\beta \in \Delta_2} W_{\beta})' \not\in G \). Hence \( A \cup B \) is \( G \)-finitistic.

**Theorem 3.3.** The join of finite number of \( G \)-finitistic \( L \)-fuzzy subsets is \( G \)-finitistic.

Proof follows from Theorem 3.2.

**Theorem 3.4.** Let \((X, \delta)\) be any \( L \)-topological Space and \( A \) be any \( L \)-fuzzy Subset of \( X \). Then \( A \) is \( G \)-finitistic subset in \((X, \delta)\) if and only if every \( L \)-fuzzy subset \( U \) of \( X \) is \( G \)-finitistic in \((X, \delta)\).

**Proof.** Suppose \( U \) is \( G \)-finitistic in \((X, \delta)\) for any \( L \)-fuzzy open subset \( U \) of \( X \). We have to show that \( A \) is \( G \)-finitistic for any \( L \)-fuzzy subset \( A \) of \( X \). Let \( \mu = \{ U_\lambda : \lambda \in \Delta \} \) be any open cover of \( A \). Then \( \bigcup_{\lambda \in \Delta} U_\lambda \) is an \( L \)-fuzzy open subset of \( X \). Therefore \( \bigcup_{\lambda \in \Delta} U_\lambda \) is a \( G \)-finitistic \( L \)-fuzzy open subset of \( X \). Since \( \mu = \{ U_\lambda : \lambda \in \Delta \} \) is also an open cover of \( \bigcup_{\lambda \in \Delta} U_\lambda \) and \( \bigcup_{\lambda \in \Delta} U_\lambda \) is a \( G \)-finitistic, there exists a finite order subfamily \( \nu = \{ V_{\alpha} : \alpha \in \Delta \} \) of \( \delta \) such that \( (\bigcup_{\alpha \in \Delta} V_{\alpha})' \not\in G \) and for each \( V_{\alpha} \in \nu \) there exists some \( U_\lambda \in \mu \) such that \( V_{\alpha} \leq U_\lambda \). Hence \( A \) is \( G \)-finitistic.

Converse is trivial.

### References


