OPTIMAL PREVIEW CONTROL FOR GENERAL TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS

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ABSTRACT: This paper examines optimal preview control issues for general time-variant discrete-time linear systems. In the case of linear variable-coefficient discrete-time systems, the difference operator is nonlinear, and for this reason, traditional preview control design methods are no longer applicable. This difficulty is overcome by the approach presented in this paper, and a corresponding augmented error system is constructed. The methodology in this paper assumed that the target signal and disturbance signal outside the preview window are all zero. For the augmented error system, we used applicable optimal control theory to obtain the optimal controller for the original system with preview feedforward compensation. Moreover, this paper employs clever matrix partition to transform the high-order Riccati equation that needs to be solved into the solution for a low-order Riccati equation. The effectiveness of the proposed methodology is demonstrated by various numerical simulations.

Keywords: time varying systems, augment error system, preview control, optimal control

1. INTRODUCTION

Model predictive control has been widely implemented in the process industry over the past 50 years due to its unique capability to systematically incorporate predictive information on future disturbances and reference signals. Analogous to but a different approach from predictive control is preview control. The preview control action is an extended feedforward control action that improves the system performance utilizing future knowledge of references and disturbances. Much research has been done on optimal preview control [1]-[5] for linear systems. Furthermore, the authors in [6]-[10] studied the robust issues of optimal preview control for linear and nonlinear systems. In recent years, researchers studied optimal preview control for systems with multiple sampling rates [11]-[15]. The system is repeatedly lifted to generate an augmented system with a single sampling rate so that standard optimal preview control can be applied. The authors in [16]-[19] studied the optimal preview control with descriptor system theory, and designed a controller with preview compensator for some descriptor causal systems.

The application of optimal preview control is ubiquitous. The authors in [20] foresee its application on heat exchanger. It has been applied to control brushless DC motors [21], vehicle suspension [22], railway suspension [23], biped locomotion [24], [25]. The authors in [26] also demonstrated the effectiveness of preview tracking control of desired trajectories in X-Y table motion control.

The problem of optimal preview control for time invariant linear systems has been studied deeply, and the research methodology had already fairly matured, which the original tracking problem is transformed to a regulation problem by augmenting the system with different operators. For the time variant linear systems, authors in [4] and [27] discussed the optimal preview control for linear time-varying discrete-time systems and continuous-time systems respectively by dynamic programming. In fact, the time invariant results were obtained as a special case of time variant formulation. In the discrete time case, enlarged state vectors may be introduced to describe the previewed future information, and the standard linear quadratic
regulator problem for time varying system may be applied for obtaining the preview control law. The same state augmentation approach does not work in the linear quadratic integral (LQI) formulation, i.e. the feedback loop includes an integral action as well as state feedback action. This is because the traditional methodology used in the linear quadratic (LQ) formulation cannot be naively extended to the LQI formulation [20]. This problem has been addressed recently in [28] by constructing the corresponding augmented system for time-varying systems, yet the input matrix of the systems is confined to constant matrix, i.e. no signal dependent.

Our work continue to study the preview control problem for time-varying discrete systems proposed in [4], which allows the state matrix, input matrix and output matrix are all time-varying matrices, and extend the results in [28] to such systems. The rest of the paper is structured as follows: Section 2 introduces the problem formulation, and the augmented system for optimal control is derived in Section 3. The optimal preview controller is designed in Section 4, and Section 5 presents simulation results. Finally, conclusions are drawn in Section 6.

2. PROBLEM

Consider the following time-varying linear system:

\[
\begin{align*}
    x(k+1) &= A(k)x(k) + B(k)u(k) + E(k)d(k) \\
    y(k) &= C(k)x(k) 
\end{align*}
\]

where \( x(k) \in R^n \) is the system state, \( u(k) \in R^r \) is the control input, \( y(k) \in R^m \) is the output of the system, \( d(k) \in R^q \) is disturbances, \( A(k), B(k), E(k), C(k) \) are time-varying dynamic matrices of \( n \times n, n \times r, n \times q, m \times n \), respectively.

Note that \( B(k) \) is time-invariant in [28], and in the current formulation, \( B(k) \) is allowed to be time-varying.

Assuming that the reference signal is \( r(k) \), \( r(k) \in R^m \), the tracking error is defined as

\[
e(k) = r(k) - y(k)
\]

In order to design the optimal preview control law for system (1), we introduce the quadratic performance index,

\[
J = \frac{1}{2} \sum_{k=1}^{N} [e^T(k)Q_e(k)e(k) + \Delta u^T(k)H \Delta u(k)]
\]

where \( Q_e, H \) are positive definite. Note that the inclusion of a quadratic term \( \Delta u(k) \) naturally introduces the integral action in the feedback loop [20].

Throughout this paper, we further assumed that:

**Assumption 1:** The preview length of disturbance is \( M_d \), i.e. the values of disturbances \( d(k-1), d(k), \ldots, d(k+M_d) \) is available, and the values of disturbances after time \( k+M_d \) is unavailable, thus set to zero.

\[
d(k+j) = 0, \quad j = M_d + 1, M_d + 2, \ldots
\]
Assumption 2: The preview length of reference signal is $M_r$, i.e. the values of the reference signals $r(k), \ldots, r(k + M_r)$ is available, and the values of the reference signals after time $k + M_r$ is unavailable, thus set to zero.

$$r(k + j) = 0, \quad j = M_r + 1, M_r + 2, \ldots$$

Note 1: At time step $k$, the disturbances at previous steps $d(k-1)$ as well as $d(k)$ are available. This is explicitly written here for clarification. It is assumed that $d(k)$ and $r(k)$ beyond the preview lengths are zero. Alternatively, they may be assumed not to change beyond the preview length: i.e. $d(k + M_d + i) = d(k + M_d), i \geq 1$, and $r(k + M_r + j) = r(k + M_r), j \geq 1$ if the disturbances and references are dominated by low frequency components. The preview control law developed in the subsequent section may be easily modified to accommodate the alternative assumption.

3. AUGMENTED SYSTEMS

In this section, we use the difference operator as conventional optimal preview control for time invariant linear systems, and reformulate the tracking problem in Section 2 into a regulator problem.

Define the difference operator as

$$\Delta x(k) = x(k) - x(k - 1)$$

In our following derivations, the following lemma is used, which is analogous to the derivative of the product of two continuous signals.

**Lemma 1:** For any matrix function with proper dimensions $G(k), v(k),$

$$\Delta [G(k)v(k)] = G(k)\Delta v(k) + \Delta G(k)v(k - 1)$$

Proof:

$$\Delta [G(k)v(k)] = G(k)v(k) - G(k - 1)v(k - 1)$$

$$= G(k)[v(k) - v(k - 1)] + [G(k) - G(k - 1)]v(k - 1)$$

$$= G(k)\Delta v(k) + \Delta G(k)v(k - 1)$$

This completes the proof.

First apply the difference operation to the system described by (1):

$$\Delta x(k + 1) = \Delta [A(k)x(k)] + \Delta [B(k)u(k)] + \Delta [E(k)d(k)]$$

$$= A(k)\Delta x(k) + \Delta A(k)x(k - 1) + B(k)\Delta u(k) + \Delta B(k)u(k - 1)$$

$$+ E(k)d(k) - E(k - 1)d(k - 1)$$

(4)

Then we apply the difference operator to $e(k + 1) = r(k + 1) - y(k + 1)$ to obtain

$$\Delta e(k + 1) = \Delta r(k + 1) - \Delta y(k + 1)$$

$$= \Delta r(k + 1) - \Delta \left[C(k + 1)x(k + 1)\right]$$
\[
= \Delta r(k+1) - C(k+1)\Delta x(k+1) - \Delta C(k+1)x(k)
\]
\[
= \Delta r(k+1) - C(k+1)\Delta x(k+1) - \Delta C(k+1)\Delta x(k) - \Delta C(k+1)x(k-1)
\]

Using \( e(k+1) = e(k) + \Delta e(k+1) \), and substituting (4) into the above expression, we have
\[
e(k+1) = e(k) - \left[ \Delta C(k+1) + C(k+1)A(k) \right]\Delta x(k) - \left[ \Delta C(k+1) + C(k+1)\Delta A(k) \right]x(k-1)
\]
\[
- C(k+1)\Delta B(k)u(k-1) - C(k+1)B(k)\Delta u(k)
\]
\[
+ C(k+1)E(k-1)d(k-1) - C(k+1)E(k)d(k) - r(k) + r(k+1)
\]

(5)

The following results can be obtained by combining (4) and (5)
\[
X_0(k+1) = \Phi(k)X_0(k) + G(k)\Delta u(k) + G_d(k)\bar{X}_d(k) + G_r(k)\bar{X}_r(k)
\]

(6)

where,

\[
X_0(k) = \begin{bmatrix} e(k) \\ \Delta x(k) \\ x(k-1) \\ u(k-1) \end{bmatrix}, \quad \bar{X}_d(k) = \begin{bmatrix} d(k-1) \\ d(k) \end{bmatrix}, \quad \bar{X}_r(k) = \begin{bmatrix} r(k) \\ r(k+1) \end{bmatrix}
\]

\[
\Phi(k) = \begin{bmatrix} I_m & -\Delta C(k+1) - C(k+1)A(k) & -\Delta C(k+1) - C(k+1)\Delta A(k) & -C(k+1)\Delta B(k) \\
0 & A(k) & \Delta A(k) & \Delta B(k) \\
0 & I_n & I_n & 0 \\
0 & 0 & 0 & I_r \end{bmatrix}
\]

\[
G(k) = \begin{bmatrix} -C(k+1)B(k) \\
B(k) \\
0 \\
I_r \end{bmatrix}, \quad G_d(k) = \begin{bmatrix} C(k+1)E(k-1) & -C(k+1)E(k) \\
-E(k-1) & E(k) \\
0 & 0 \end{bmatrix} = \begin{bmatrix} G_{d1}(k) & G_{d2}(k) \end{bmatrix}
\]

\[
G_r = \begin{bmatrix} -I_m & I_m \\
0 & 0 \\
0 & 0 \end{bmatrix} = [G_{r1} -G_{r1}]
\]

Note 2: The system state \( X_0(k) \) is augmented to include \( x(k-1) \). In addition, we directly employ \( d(k-1), d(k), r(k), \) and \( r(k+1) \) (instead of their differences) to construct the augmented error system. The development of augmented equation above allows us to extend the previous method [3], [5], [11]-[20] to time-varying systems. Also, the results in [28] are naturally extended to time-varying linear systems.
For system (6), the quadratic performance index (3) can be rewritten as
\[
J = \frac{1}{2} \sum_{k=1}^{N} \left[ X_0^T(k)QX_0(k) + \Delta u^T(k)H\Delta u(k) \right]
\]
where \( Q = \begin{bmatrix} Q_e & 0 \\ 0 & 0 \end{bmatrix} \) is positive semi-definite.

According to Assumptions 1, and 2, at time instance \( k \), the disturbances at \( k-1, k, \ldots, k + M_d \), i.e. \( d(k-1), d(k), \ldots, d(k + M_d) \), are known and the future values of the disturbance beyond \( k + M_d \) are zero. The reference signals \( r(k), \ldots, r(k + M_r) \) are readily defined, and the future values of the reference signals after time step \( k + M_r \) are zero. All known information about disturbances and references at time \( k \) is conveniently packaged as
\[
X_d(k+1) = A_d X_d(k) \tag{7}
\]
\[
X_r(k+1) = A_r X_r(k) \tag{8}
\]
where,
\[
A_d = \begin{bmatrix} 0 & I_q & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_q \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad X_d(k) = \begin{bmatrix} d(k-1) \\ d(k) \\ \vdots \\ d(k + M_d) \end{bmatrix} = \begin{bmatrix} X_d(k) \\ X_d(k+1) \\ \vdots \\ X_d(k + M_d) \end{bmatrix};
\]
\[
A_r = \begin{bmatrix} 0 & I_m & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I_m \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \quad X_r(k) = \begin{bmatrix} r(k) \\ r(k+1) \\ \vdots \\ r(k + M_r) \end{bmatrix} = \begin{bmatrix} X_r(k) \\ X_r(k+1) \\ \vdots \\ X_r(k + M_r) \end{bmatrix}.
\]

Note 3: If the disturbances and references do not change beyond the preview length (recall Note 1), the right bottom block of \( A_d \) and \( A_r \) become \( I_m \) and \( I_q \). Define
\[
G_{rd}(k) = \begin{bmatrix} G_d(k) & 0 & \cdots & 0 \end{bmatrix}, \quad G_{rr} = \begin{bmatrix} G_r & 0 & \cdots & 0 \end{bmatrix}
\]
The resultant augmented system is obtained by combining equations (7), (8), and (6)
\[
X_p(k+1) = \Phi_p(k) X_p(k) + G_p(k) \Delta u(k) \tag{9}
\]
where,

\[
X_p(k) = \begin{bmatrix} X_0(k) \\ X_d(k) \\ X_r(k) \end{bmatrix},
\Phi_p(k) = \begin{bmatrix} \Phi(k) & G_{pd}(k) & G_p \end{bmatrix},
G_p(k) = \begin{bmatrix} G(k) \\ 0 \\ 0 \end{bmatrix}
\]

The performance index (3) now can be further rewritten as:

\[
J = \frac{1}{2} \sum_{k=1}^{N} \left[ X_p^T(k) \tilde{Q} X_p(k) + \Delta u^T(k) H \Delta u(k) \right]
\]

(10)

where,

\[
\tilde{Q} = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Note that the original tracking problem has been simplified to a regulator problem to find the optimal control input \( \Delta u(k) \) for system (9) with the performance index defined by (10).

**4. DESIGN OF OPTIMAL CONTROLLER WITH PREVIEW ACTION**

In this section, the optimal control problem for system (9) is solved, then the optimal preview controller for the original system (1) is derived by simplifying Riccati equations for system (9).

**4.1. Optimal Control for Augmented System**

According to the fundamental results in optimal control theory [30], the following result is readily available.

Theorem 1: The optimal control input for system (9) that minimizes index function (10) is

\[
\Delta u^*(k) = -\left[ H + G_p^T(k) P(k+1) G_p(k) \right]^{-1} G_p^T(k) P(k+1) \Phi_p(k) X_p(k)
\]

(11)

where, the positive semi-definite matrix \( P(k) \) is the solution to the following Riccati equation:

\[
\begin{cases}
P(k) = \tilde{Q} + \Phi_p^T(k) P(k+1) \Phi_p(k) - \Phi_p^T(k) P(k+1) G_p(k) \\
\times \left[ H + G_p^T(k) P(k+1) G_p(k) \right]^{-1} G_p^T(k) P(k+1) \Phi_p(k) \\
P(N+1) = 0
\end{cases}
\]

(12)

**4.2. Decomposition of Riccati Equation**

The optimal preview control problem for system (9) has been addressed in the previous section. However, it would be computationally prohibitive to solve Riccati equation (12) for long preview length \( M_r \) and \( M_d \) as \( P(k) \) and \( \Phi_p(k) \) are matrices of \( [2n+r+(M_r+2)m+(M_d+2)q] \times [2n+r+(M_r+2)m+(M_d+2)q] \). This complexity can be resolved by a similar approach in [4, 28] to partition the matrices in (12) into blocks with lower dimension so that the solution to (12) can be obtained by solving a smaller Riccati equation.
Noticing the special structure of \( G_p(k) \), \( \Phi_p(k) \), and \( \tilde{Q}, P(k) \) can be partitioned as:

\[
P(k) = \begin{bmatrix} P_0(k) & W_1(k) & W_2(k) \\ W_1^T(k) & Z(k) & X(k) \\ W_2^T(k) & X^T(k) & Y(k) \end{bmatrix}.
\]

The matrices are partitioned in a way that equation (12) can be solved block by block. The Riccati equation (12) can be reformulated with partitioned \( P(k) \):

\[
\begin{bmatrix} P_0(k) & W_1(k) & W_2(k) \\ W_1^T(k) & Z(k) & X(k) \\ W_2^T(k) & X^T(k) & Y(k) \end{bmatrix} = \begin{bmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \Phi^T(k)P_0(k+1)\Phi(k) & \Phi^T(k)[P_0(k+1)G_{pe}(k)+W_1(k+1)A_d] & \Phi^T(k)[P_0(k+1)G_{pr}+W_2(k+1)A_d] \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix}
\]

\[
\begin{bmatrix} \Phi^T(k)P_0(k+1)G(k) \\ \ast \\ \ast \end{bmatrix} \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1}
\]

\[
\times \begin{bmatrix} G^T(k)P_0(k+1)\Phi(k) & G^T(k)[P_0(k+1)G_{pe}(k)+W_1(k+1)A_d] & G^T(k)[P_0(k+1)G_{pr}+W_2(k+1)A_d] \end{bmatrix}
\]

where, \( \ast \) denotes the irrelevant block matrices in (11). Based on symmetry of (13), the equation (13) is equivalent to the following 3 equations.

\[
\begin{aligned}
P_0(k) &= Q + \Phi^T(k)P_0(k+1)\Phi(k) - \Phi^T(k)P_0(k+1)G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k)P_0(k+1)\Phi(k) \\
W_1(k) &= \Phi^T(k) \left[ I - P_0(k+1)G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k) \right] [P_0(k+1)G_{pe} + W_1(k+1)A_d] \\
W_2(k) &= \Phi^T(k) \left[ I - P_0(k+1)G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k) \right] [P_0(k+1)G_{pr} + W_2(k+1)A_d]
\end{aligned}
\]

Note that the first equation in (14) is a Riccati equation of lower dimension \((m+2n+r) \times (m+2n+r)\).

\[
P_0(k) = Q + \Phi^T(k)P_0(k+1)\Phi(k) - \Phi^T(k)P_0(k+1)G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k)P_0(k+1)\Phi(k)
\]

(15)

The terminal condition is

\[
P_0(N+1) = 0
\]

Define

\[
\Xi(k) = \left[ I - G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k)P_0(k+1) \right] \Phi(k)
\]

The last two equations in (14) are reformulated as:
\[
\begin{align*}
W_1(k) &= \Xi^T(k) \left[ P_0(k+1)G_{p_d}(k) + W_1(k+1)A_d \right] \\
W_2(k) &= \Xi^T(k) \left[ P_0(k+1)G_{p_r} + W_2(k+1)A_r \right]
\end{align*}
\] (16)

First, \( W_1(k) \) can be solved by the first equation in (16). Because of the special structure of \( A_d \) that it consists of \((M_d + 2) \times (M_d + 2)\) block matrices of \(q \times q\), \( W_1(k) \) can be partitioned as

\[
W_1(k) = \begin{bmatrix} W_1^{(1)}(k) & W_1^{(2)}(k) & \cdots & W_1^{(M_d+1)}(k) & W_1^{(M_d+2)}(k) \end{bmatrix}
\]

Thus,

\[
\begin{bmatrix} W_1^{(1)}(k) & W_1^{(2)}(k) & \cdots & W_1^{(M_d+1)}(k) & W_1^{(M_d+2)}(k) \end{bmatrix} = \Xi^T(k) \left[ P_0(k+1)G_{d_1}(k) + W_1^{(1)}(k+1) + W_1^{(2)}(k+1) + \cdots + W_1^{(M_d)}(k+1) \right]
\]

As a result,

\[
\begin{align*}
W_1^{(1)}(k) &= \Xi^T(k)P_0(k+1)G_{d_1}(k) \\
W_1^{(2)}(k) &= \Xi^T(k) \left[ P_0(k+1)G_{d_2}(k) + W_1^{(1)}(k+1) \right] \\
W_1^{(3)}(k) &= \Xi^T(k)W_1^{(2)}(k+1) \\
&\vdots \\
W_1^{(M_d+1)}(k) &= \Xi^T(k)W_1^{(M_d)}(k+1) \\
W_1^{(M_d+2)}(k) &= \Xi^T(k)W_1^{(M_d+1)}(k+1)
\end{align*}
\] (17)

Similarly, \( W_2(k) \) can be solved by partitioning

\[
W_2(k) = \begin{bmatrix} W_2^{(1)}(k) & W_2^{(2)}(k) & \cdots & W_2^{(M_r)}(k) & W_2^{(M_r+1)}(k) \end{bmatrix}
\]

Thus,

\[
\begin{align*}
W_2^{(1)}(k) &= \Xi^T(k)P_0(k+1)G_{r_1} \\
W_2^{(2)}(k) &= \Xi^T(k) \left[ -P_0(k+1)G_{r_1} + W_2^{(1)}(k+1) \right] \\
W_2^{(3)}(k) &= \Xi^T(k)W_2^{(2)}(k+1) \\
&\vdots \\
W_2^{(M_r)}(k) &= \Xi^T(k)W_2^{(M_r-1)}(k+1) \\
W_2^{(M_r+1)}(k) &= \Xi^T(k)W_2^{(M_r)}(k+1)
\end{align*}
\] (18)

The terminal condition can be simply derived from (12),
\[ W_1(N+1) = 0, \quad W_2(N+1) = 0 \]

Or equivalently,
\[ W_1^{(i)}(N+1) = 0 \quad (i = 1, 2, \cdots, M_d + 2) \]
\[ W_2^{(i)}(N+1) = 0 \quad (i = 1, 2, \cdots, M_r + 1) \]

Consequently,
\[ W_1(k) = \begin{bmatrix} W_1^{(1)}(k) & W_1^{(2)}(k) & \cdots & W_1^{(M_d+1)}(k) & W_1^{(M_d+2)}(k) \end{bmatrix} \quad (k = N, N-1, \cdots, 2, 1) \]
and
\[ W_2(k) = \begin{bmatrix} W_2^{(1)}(k) & W_2^{(2)}(k) & \cdots & W_2^{(M_r)}(k) & W_2^{(M_r+1)}(k) \end{bmatrix} \quad (k = N, N-1, \cdots, 2, 1) \]
can be readily solved by (17) and (18) iteratively.

Note that the boundary condition for (15) can be derived similarly from (12).
\[ P_0(N+1) = 0 \]

The optimal control input (11) becomes
\[
\Delta u^*(k) = - \left[ H + G^T(k) P_0(k+1) G(k) \right]^{-1} G^T(k) \{ P_0(k+1) \Phi(k) X_0(k) \\
+ [ P_0(k+1) G_{p_2}(k) + W_1(k+1) A_d ] X_d(k) + [ P_0(k+1) G_{p_2} + W_2(k+1) A_r ] X_r(k) \} \\
= - \left[ H + G^T(k) P_0(k+1) G(k) \right]^{-1} G^T(k) \{ P_0(k+1) \Phi(k) X_0(k) \\
+ P_0(k+1) \left[ G_{d_1}(k) d(k-1) + G_{d_2}(k) d(k) \right] + \sum_{i=1}^{M_r+1} W_1^{(i)}(k+1) d(k+i-1) \\
+ P_0(k+1) G_{r_1} \left[ r(k) - r(k+1) \right] + \sum_{i=1}^{M_r} W_2^{(i)}(k+1) r(k+i) \} \]
\[ 1 \leq k \leq N \] (19)

where, \( P_0(k+1) \) is solved by lower dimension Riccati equation (15), and \( W_1^{(i)}(k+1), W_2^{(i)}(k+1) \) are calculated by (17) and (18), respectively.

4.3. Optimal Feed Forward Preview Control
In this section, the optimal feedforward preview control for system (1) is presented. Actually \( u(k) \) is readily obtained from equation (19), by noting that \( \Delta u(k) = u(k) - u(k-1) \):

Theorem 2: the optimal control input \( u(k) \) for system (1) that minimizes the performance index (10) is given by:
\[ u(k) = u(k-1) - \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k) \{ P_0(k+1)\Phi(k)X_0(k) \\
+ P_0(k+1)[G_{d_1}(k)d(k-1) + G_{d_2}(k)d(k)] + \sum_{i=1}^{M+1} W_1^{(i)}(k+1)d(k+i-1) \\
+ P_0(k+1)G_{r_1}[r(k)-r(k+1)] + \sum_{i=1}^{M} W_2^{(i)}(k+1)r(k+i) \} \]

\[ 1 \leq k \leq N \quad (20) \]

where \( P_0(k) \) and other parameters are determined by the following equations:

\[
\begin{align*}
P_0(k) &= Q + \Phi^T(k)P_0(k+1)\Phi(k) - \Phi^T(k)P_0(k+1)G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k)P_0(k+1)\Phi(k) \\
\begin{align*}
P_0(N+1) &= 0 \\
\Xi(k) &= \left\{ I - G(k) \left[ H + G^T(k)P_0(k+1)G(k) \right]^{-1} G^T(k)P_0(k+1) \right\} \Phi(k) \\
W_1^{(1)}(k) &= \Xi^T(k)P_0(k+1)G_{d_1}(k) \\
W_1^{(2)}(k) &= \Xi^T(k)\left[ P_0(k+1)G_{d_2}(k) + W_1^{(1)}(k+1) \right] \\
W_1^{(3)}(k) &= \Xi^T(k)W_1^{(2)}(k+1) \\
&\vdots \\
W_1^{(M+1)}(k) &= \Xi^T(k)W_1^{(M)}(k+1) \\
W_1^{(M+2)}(k) &= \Xi^T(k)W_1^{(M+1)}(k+1) \\
W_1^{(i)}(N+1) &= 0 \quad (i = 1, 2, \ldots, M + 2) \\
\end{align*}
\]

\[
\begin{align*}
W_2^{(1)}(k) &= \Xi^T(k)P_0(k+1)G_{r_1} \\
W_2^{(2)}(k) &= \Xi^T(k)\left[ -P_0(k+1)G_{r_1} + W_2^{(1)}(k+1) \right] \\
W_2^{(3)}(k) &= \Xi^T(k)W_2^{(2)}(k+1) \\
&\vdots \\
W_2^{(M-1)}(k) &= \Xi^T(k)W_2^{(M-2)}(k+1) \\
W_2^{(M)}(k) &= \Xi^T(k)W_2^{(M-1)}(k+1) \\
W_2^{(i)}(N+1) &= 0 \quad (i = 1, 2, \ldots, M + 1) \\
\end{align*}
\]

\( X_0(k) \) is determined by:

\[
X_0(k) = \begin{bmatrix} e(k) \\ \Delta x(k) \\ x(k-1) \\ u(k-1) \end{bmatrix} = \begin{bmatrix} r(k) - y(k) \\ x(k) - x(k-1) \\ x(k-1) \\ u(k-1) \end{bmatrix}
\]
The initial state $x(0)$ and control input $u(0)$ in $X_0(1) = \begin{bmatrix} e(1) \\ x(1) - x(0) \\ x(0) \\ u(0) \end{bmatrix}$ are specified as required.

Note 4: Let $F = -\left[ H + G^T(k)P_0(k + 1)G(k) \right]^{-1}$, and consider the optimal control solution (20). The second term $FG^T(k)P_0(k + 1)\Phi(k)X_0(k)$ includes state feedback and integrator; the third term in (20) $FG^T(k)P_0(k + 1)\left[ G_{d1}(k)d(k - 1) + G_{d2}(k)d(k) \right]$ is derived from the disturbance of the current step and the previous step; the fourth term in (20) $FG^T(k)\left( \sum_{i=1}^{M+1} W_1^{(i)}(k + 1)d(k + i - 1) \right)$ is the feedforward control based on the disturbance preview; the fifth term $FG^T(k)P_0(k + 1)G_{r1}(r(k) - r(k + 1))$ is the control action to capture reference signal change in the next time step; the last term $FG^T(k)\left( \sum_{i=1}^{M_n} W_2^{(i)}(k + 1)r(k + i) \right)$ is the feedforward compensation based on the reference signals preview.

5. SIMULATION RESULTS

In order to make a comparison of preview control with different preview lengths, a numerical example is introduced. We note that based on optimal control theory and the previous method of augmented error system, the optimal control input $u(k)$ with no preview for system (1) is obtained as:

$$u(k) = u(k - 1) - \left[ H + G^T(k)P_0(k + 1)G(k) \right]^{-1} G^T(k)P_0(k + 1)\Phi(k) \begin{bmatrix} r(k) - y(k) \\ x(k) - x(k - 1) \\ x(k - 1) \\ u(k - 1) \end{bmatrix}$$

where all matrices are as described in Theorem 2.

Example 1: Consider the following time-varying linear system:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.24 + \frac{1}{1+k^2} & 0.01 + e^{-k^2} \\ 0.04 + 0.2\sin k & -0.35 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.027k + 1 \\ 0.12 \end{bmatrix} u(k) + \begin{bmatrix} 2 \\ k \end{bmatrix} d(k)$$

$$y(k) = \begin{bmatrix} 0.45 \\ -0.001\cos(2\pi k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Note that $A(k) = \begin{bmatrix} -0.24 + \frac{1}{1+k^2} & 0.01 + e^{-k^2} \\ 0.04 + 0.2\sin k & -0.35 \end{bmatrix}, \quad B(k) = \begin{bmatrix} 0.027k + 1 \\ 0.12 \end{bmatrix}, \quad E(k) = \begin{bmatrix} 2 \\ k \end{bmatrix}$, and
$C(k) = \begin{bmatrix} 0.45 & -0.001 \cos(2\pi k) \end{bmatrix}$. The parameters in the performance index (10) are defined as $Q_e = 100, H = 5$.

Note that the coefficient matrix of disturbance $d(k)$ is $E(k) = \begin{bmatrix} 2 \\ k \end{bmatrix}$, so if $d(k) \neq 0$, the disturbance effect is unlimited.

For the following types of reference signals and disturbances, numerical simulations show that the output response of closed-loop system without preview information fails to track the reference signals, but in the case of controller with preview action, the desired reference signals are successfully tracked.

In the following, we choose the initial state $x(0)$ as $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}$.

(1) Step reference signal and step disturbance

Let the reference signals be

$$r(k) = \begin{cases} 0.6, & k > 50 \\ 0, & k \leq 50 \end{cases}$$  \hspace{1cm} (22)

and the disturbances

$$d(k) = \begin{cases} 1.3, & k > 75 \\ 0, & k \leq 75 \end{cases}$$  \hspace{1cm} (23)

For two sets of simulations, the preview lengths are set to $M_r = 20, M_d = 10$ and $M_r = 2, M_d = 1$, respectively.

![Figure 1: The Output Response of System (21) to Step Reference Signal and Step Disturbance](image-url)
The system response is plotted in Figure 1. It is observed that the desired reference signals are all successfully tracked. In order to examine the effect of the preview length, we calculated the tracking errors in both cases, which are shown in Figure 2.

![Figure 2: The Tracking Errors about Step Reference Signal and Step Disturbance](image)

As shown in Figure 2, the increase of preview length contributes to reduce overshoot and shorter adjustment time.

(2) Step reference signals and periodic disturbances

The step reference signals are defined by (22), and the disturbances is

$$d(k) = 0.3 \cos(0.2k)$$

we also choose two sets of preview lengths: $M_r = 20, M_d = 10$ and $M_r = 2, M_d = 1$.

The tracking results shown in Figure 3 indicate that the proposed controller manages to handle periodic disturbances. The corresponding tracking errors are shown in Figure 4, we can also clearly see that the controller with longer preview lengths is more effective in reducing the tracking error.

(3) Periodic reference signals and step disturbance

The reference signal is

$$r(k) = 0.6 \sin(0.1k)$$
Figure 3: The Output Response of System (21) to Step Reference Signal and Periodic Disturbance

Figure 4: The Tracking Errors about Step Reference Signal and Periodic Disturbance
and the disturbance

\[
d(k) = \begin{cases} 
1.3, & k > 75 \\
0, & k \leq 75 
\end{cases}
\]

Two sets of preview lengths are \( M_r = 20, M_d = 10 \) and \( M_r = 2, M_d = 1 \).

![Figure 5: System (21) Output Response to Periodic Reference Signal and Step Disturbance](image)

Not surprisingly, the controller successfully tracks the periodic reference signals in Figure 5. The tracking errors are shown in Figure 6, and Figure 7 is a partially enlarged view of Figure 6.

(4) Periodic reference signals and periodic disturbances

Let the reference signals and disturbances be

\[
r(k) = 0.6 \sin \left( \frac{k}{3} \right)
\]

\[
d(k) = 0.3 \cos(0.2k)
\]

Note that their frequencies are different (Figure 8).
Figure 6: The Tracking Errors about Periodic Reference Signal and Step Disturbance

Figure 7: The Partially Enlarged View of Figure 6
Let two sets of preview lengths are $M_r = 20, M_d = 10$ and $M_r = 2, M_d = 1$.

The simulation results in Figure 9 show that the controller successfully tracks periodic reference signals with disturbances of different frequency. The error signals are shown in Figure 10.
(5) General reference signals with increased amplitude and step disturbances

The reference signals are as given by (23), and the disturbances are

\[ r(k) = 2.4\sqrt{k} \cos \left( 0.05k - \frac{\pi}{3} \right) \]  \hspace{1cm} (28)

The amplitude of the reference signals increases as time increases (Figure 11).

Two sets of preview lengths are \( M_r = 20, M_d = 10 \) and \( M_r = 2, M_d = 1 \). The tracking results are shown in Figure 12.

The tracking errors are shown in Figure 13, and Figure 14 is a partially enlarged view of Figure 13. Note that the controller with longer preview lengths significantly reduces the tracking errors.

### 6. CONCLUSIONS

In this work, the optimal preview control with integral action for time-varying discrete-time linear systems is studied. We extended and improved the results of optimal preview control design for time-invariant systems so that the methodology can be applied to time-varying systems.

A procedure is devised to design an optimal feedforward preview controller by augmenting the original time-varying discrete-time system, and explicitly solving a finite time optimal control problem. It was
Figure 11: Reference Signals Profile (28)

Figure 12: System (21) Output Response to General Reference Signals and Step Disturbances
Figure 13: The Tracking Errors about General Reference Signals and Step Disturbances

Figure 14: The Partially Enlarged View of Figure 13
assumed that the disturbance and reference input do not change beyond the preview length, but the result can be modified to accommodate different assumptions. A series of numerical simulations is presented to demonstrate the effectiveness of the proposed optimal feedforward preview controller design.

REFERENCES


