ON THE JACOBSTHAL AND JACOBSTHAL-LUCAS SEQUENCES BY RECURRENCE RELATIONS

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In this paper, we give another proofs of two well-known results relative to the Jacobsthal and Jacobsthal-Lucas sequences and revise one of them. Moreover, we establish two equalities between this two sequences and referred closed-form sequences.

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1. INTRODUCTION

The Jacobsthal and Jacobsthal-Lucas sequences [2] are defined by the following recurrence relations:

\[ J_{n+2} = J_{n+1} + 2J_n, \quad J_0 = 0, \quad J_1 = 1, \]

and

\[ j_{n+2} = j_{n+1} + 2j_n, \quad j_0 = 0, \quad j_1 = 1, \]

for \( n \geq 0 \). The first nine values of these two sequences are 1, 1, 3, 5, 11, 21, 43, 85, 171 and 1, 5, 7, 17, 31, 65, 127, 257, 511, respectively. The permanent of an n-square matrix is similar to the determinant but all of the signs used in the Laplace expansion of minors are positive and is defined by

\[ \text{per} \ A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{\sigma(i)}, \]

where the summation extends over all permutations \( \sigma \) of the symmetric group \( S_n \) [1]. In a recent paper [2], the authors defined following matrices:

\[ H_n = \begin{bmatrix}
3 & 2 & & & \\
1 & 1 & 2 & & 0 \\
1 & 1 & 2 & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
0 & 1 & 1 & 2 & 1 & 1
\end{bmatrix}, \]

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and

\[
K_n = \begin{bmatrix}
1 & 2 \\
1 & 3 & 2 & 0 \\
1 & 1 & 2 \\
& \ddots & \ddots & \ddots \\
1 & 1 & 2 \\
0 & 1 & 1 & 2 \\
& & & 1 & 1
\end{bmatrix}
\]

Then they gave Hadamard product of the matrices by

\[
S = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
-1 & 1 & \cdots & 1 & 1 \\
1 & -1 & \cdots & 1 & 1 \\
& \ddots & \ddots & \ddots & \ddots \\
1 & 1 & \cdots & -1 & 1
\end{bmatrix}
\]

So they obtained

\[
A_n = \begin{bmatrix}
3 & 2 \\
-1 & 1 & 2 & 0 \\
-1 & 1 & 2 \\
& \ddots & \ddots & \ddots \\
0 & -1 & 1 & 2 \\
& & & -1 & 1
\end{bmatrix}
\]

and

\[
B_n = \begin{bmatrix}
1 & 2 \\
-1 & 3 & 2 & 0 \\
-1 & 1 & 2 \\
& \ddots & \ddots & \ddots \\
0 & -1 & 1 & 2 \\
& & & -1 & 1
\end{bmatrix}
\]

By using the technique of contraction, they derived the following two identities:

\[
\det (A_n) = \per H_n = J_n
\]
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\[ \det(B_n) = \text{per } K_n = j_n \]  

(2)

Note that per \( H_n \) means the permanent of \( H_n \).

In this paper, we will prove the above two identities by using recurrence relations and modify the result of (1).

2. ALTERNATING PROOFS OF TWO DETERMINANTS RELATIVE TO THE JACOBSTHAL AND JACOBSTHAL–LUCAS NUMBERS

In a recent paper [2], the authors derived identities (1) and (2) by using contraction. Now we are in a position to modify identity (1) and give another proof of identity (2) by recurrence relation.

In order to prove Theorem 2, we establish Lemma 1 at first.

Lemma 1:

\[
\sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k-l} \times 2^l + \sum_{l=2}^{\left\lfloor \frac{k+1}{2} \right\rfloor} C_{l}^{k-l-1} \times 2^{l+1} = \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l.
\]

Proof: (1) The case \( k \) is odd:

Then \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor \) and \( \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor + 1 = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \). So

\[
\sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k-l} \times 2^l + \sum_{l=2}^{\left\lfloor \frac{k+1}{2} \right\rfloor} C_{l}^{k-l-1} \times 2^{l+1} = \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k-l} \times 2^l + \sum_{m=3}^{\left\lfloor \frac{k+1}{2} \right\rfloor} C_{m}^{k-m} \times 2^m
\]

\[
= \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l + C_{\left\lfloor \frac{k}{2} \right\rfloor}^{k-\left\lfloor \frac{k}{2} \right\rfloor-1} \times 2^{\left\lfloor \frac{k}{2} \right\rfloor+1}
\]

\[
= \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l + C_{\left\lfloor \frac{k}{2} \right\rfloor}^{k-1-\left\lfloor \frac{k}{2} \right\rfloor} \times 2^{\left\lfloor \frac{k}{2} \right\rfloor+1}
\]

\[
= \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l.
\]

(2) The case \( k \) is even:

Then \( \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \) and \( \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{k-1}{2} \right\rfloor + 1 \). So

\[
\sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k-l} \times 2^l + \sum_{l=2}^{\left\lfloor \frac{k+1}{2} \right\rfloor} C_{l}^{k-l-1} \times 2^{l+1} = \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k-l} \times 2^l + \sum_{m=3}^{\left\lfloor \frac{k+1}{2} \right\rfloor} C_{m}^{k-m} \times 2^m
\]

\[
= \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l
\]

\[
= \sum_{l=3}^{\left\lfloor \frac{k}{2} \right\rfloor} C_{l}^{k+1-l} \times 2^l.
\]
Theorem 2:

\[
\det T_n = 3 + C_1^{n-2} \times 2 + C_2^{n-2} \times 2^2 + \sum_{l=3}^{n} C_l^{n-l} \times 2^l.
\]

Proof: We use mathematical induction on \( n \), the size of \( T_n \).

(1) The case \( n = 2 \):

\[
\det T_2 = \det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = 3.
\]

(2) The case \( k \Rightarrow k + 1 \) with \( k > 1 \): Assuming

\[
\det T_k = 3 + C_1^{k-2} \times 2 + C_2^{k-2} \times 2^2 + \sum_{l=3}^{k} C_l^{k-l} \times 2^l
\]

holds, we want to prove

\[
\det T_{k+1} = 3 + C_1^{k-1} \times 2 + C_2^{k-1} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^l.
\]

\[
\det T_{k+1} = \det T_k + 2 \det T_{k-1}
\]

\[
= 3 + C_1^{k-2} \times 2 + C_2^{k-2} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^l
\]

\[
+ 2 \times \left[ 3 + C_1^{k-3} \times 2 + C_2^{k-3} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^l \right]
\]

\[
= 3 + C_1^{k-2} \times 2 + C_2^{k-2} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^l
\]

\[
+ 6 + C_1^{k-3} \times 2^2 + C_2^{k-3} \times 2^3 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^{l+1}
\]

\[
= 3 + C_1^{k-1} \times 2 + C_2^{k-2} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k-1} \times 2^l + C_1^{k-2} \times 2^2 + \sum_{l=2}^{k+1} C_l^{k-1-l} \times 2^{l+1}
\]

\[
= 3 + C_1^{k-1} \times 2 + C_2^{k-2} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k-l} \times 2^l + \sum_{l=2}^{k+1} C_l^{k-1-l} \times 2^{l+1}
\]

\[
= 3 + C_1^{k-1} \times 2 + C_2^{k-1} \times 2^2 + \sum_{l=3}^{k+1} C_l^{k+1-l} \times 2^l.
\]

So if \( n = k + 1 \), the theorem holds. This completes the proof.
Lemma 3: Let

\[
T_n = \begin{bmatrix}
1 & 2 & 0 \\
-1 & 1 & 2 \\
0 & -1 & 1 & 2 \\
\vdots & & & \\
-1 & 1 & 2 \\
-1 & 1 & 2 \\
\end{bmatrix}
\]

be a \( n \times n \) matrix, then

\[
\det T_n = \det T_{n-1} + 2 \det T_{n-2}.
\]

Proof:

\[
\det T_n = \det \begin{bmatrix}
1 & 2 & 0 \\
-1 & 1 & 2 \\
0 & -1 & 1 & 2 \\
\vdots & & & \\
-1 & 1 & 2 \\
-1 & 1 & 2 \\
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
1 & 2 & 0 \\
-1 & 1 & 2 \\
0 & -1 & 1 & 2 \\
\vdots & & & \\
-1 & 1 & 2 \\
-1 & 1 & 2 \\
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
1 & 2 & 0 \\
-1 & 1 & 2 \\
0 & -1 & 1 & 2 \\
\vdots & & & \\
-1 & 1 & 2 \\
-1 & 1 & 2 \\
\end{bmatrix} + 2 \det \begin{bmatrix}
1 & 2 & 0 \\
-1 & 1 & 2 \\
0 & -1 & 1 & 2 \\
\vdots & & & \\
-1 & 1 & 2 \\
-1 & 1 & 2 \\
\end{bmatrix}
\]

\[
= \det T_{n-1} + 2 \det T_{n-2}.
\]

Lemma 4:

\[
\det A_n = 3 \det T_{n-1} + 2 \det T_{n-2}.
\]
Proof:

\[
\begin{bmatrix}
3 & 2 \\
-1 & 1 & 2 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -1 & 1 & 2 \\
-1 & 1 \\
\end{bmatrix}
\]

\[
\det A_n = \det \begin{bmatrix}
3 & 2 \\
-1 & 1 & 2 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -1 & 1 & 2 \\
-1 & 1 \\
\end{bmatrix} = 3 \det \begin{bmatrix}
1 & 2 \\
-1 & 1 & 2 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -1 & 1 & 2 \\
-1 & 1 \\
\end{bmatrix} - (-1)^2 \det \begin{bmatrix}
2 & 1 & 2 & 0 \\
-1 & 1 & 2 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & -1 & 1 & 2 \\
-1 & 1 \\
\end{bmatrix} 
\]

\[= 3 \det T_{n-1} + 2 \det T_{n-2}.
\]

**Theorem 5:**

\[\det A_n = \text{per } H_n = J_{n+2}.
\]

**Proof:** We use mathematical induction on \(n\), the size of \(A_n\).

1. The case \(n = 2\):

\[\det A_2 = \det \begin{bmatrix}
3 & 2 \\
-1 & 1 \\
\end{bmatrix} = 5 = J_4.
\]

2. The case \(k \Rightarrow k + 1\) with \(k > 1\): Assuming \(\det A_k = \text{per } H_k = J_{k+2}\) holds, we want to prove \(\det A_{k+1} = \text{per } H_{k+1} = J_{k+3}\). Because

\[
\begin{align*}
\det A_{k+1} &= 3 \det T_k + 2 \det T_{k-1} \\
&= 3 (\det T_{k-1} + 2 \det T_{k-2}) + 2 (\det T_{k-2} + 2 \det T_{k-3}) \\
&= 3 \det T_{k-1} + 6 \det T_{k-2} + 2 \det T_{k-3} + 4 \det T_{k-3} \\
&= 3 \det T_{k-1} + 2 \det T_{k-2} + 6 \det T_{k-2} + 4 \det T_{k-3} \\
&= \det A_k + 2 \det A_{k-1} \\
&= J_{k+2} + 2 J_{k+1} \\
&= J_{k+3}.
\end{align*}
\]
So if $n = k + 1$, the theorem holds. This completes the proof.

**Corollary 6:**

\[
\det A_n = \text{per } H_n = J_{n+2}
\]

\[
= 3 \left\{ 3 + C_1^{n-3} \times 2 + C_2^{n-3} \times 2^2 + \sum_{l=3}^{n-1} C_l^{n-1-l} \times 2^l \right\} + 2 \left\{ 3 + C_1^{n-4} \times 2 + C_2^{n-4} \times 2^2 + \sum_{l=3}^{n-2} C_l^{n-2-l} \times 2^l \right\}.
\]

**Lemma 7:**

\[
\det B_n = \det T_{n-1} + 4 \det T_{n-2}.
\]

**Proof:**

\[
\det B_n = \det \begin{bmatrix}
1 & 2 & 0 \\
-1 & 3 & 2 & 0 \\
-1 & 1 & 2 & 0 \\
0 & -1 & 1 & 2 \\
-1 & 1 & 1 & 1
\end{bmatrix}
\]

\[
= \det \begin{bmatrix}
1 & 2 & 0 \\
-2 & 1 & 2 & 0 \\
-1 & 1 & 2 & 0 \\
0 & -1 & 1 & 2 \\
-1 & 1 & 1 & 1
\end{bmatrix}^{(n-1)\times(n-1)}
\]

\[
-(-2) \times \det \begin{bmatrix}
2 & 0 \\
-1 & 2 & 0 \\
-1 & 1 & 2 & 0 \\
0 & -1 & 1 & 2 \\
-1 & 1 & 1 & 1
\end{bmatrix}^{(n-1)\times(n-1)}
\]
Theorem 8:

$$\det B_n = \text{per} K_n = j_n.$$  

Proof: We use mathematical induction on $n$, the size of $B_n$.

(1) The case $n = 2$.

$$\det B_2 = \det \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = 5 = j_2.$$  

(2) The case $k = k + 1$ with $k > 1$. Assuming $\det B_k = \text{per} K_k = j_k$ holds, we want to prove $\det B_{k+1} = \text{per} K_{k+1} = j_{k+1}$.

Because $\det B_{k+1} = \det T_{k+1} + 4 \det T_k - 4 \det T_{k-1}$

$$= \det T_k + 4 \det T_{k-1} + 4 \det T_{k-2} + 4 \det T_{k-3} - 4 \det T_{k-2} = \det T_k + 4 \det T_{k-1} + 4 \det T_{k-2} + 4 \det T_{k-3}.$$  

We have $\det T_k = \det T_k - 4 \det T_{k-1} + 4 \det T_{k-2} - 4 \det T_{k-3} + 2 \det T_{k-2} + 2 \det T_{k-3}$

$$= \det T_k - 4 \det T_{k-1} + 2 \det T_{k-2} + 2 \det T_{k-3} = \det T_k + 4 \det T_{k-1} + 4 \det T_{k-2} + 4 \det T_{k-3}.$$  

So if $n = k + 1$, the theorem holds. This completes the proof.

Corollary 9:

$$\det B_n = \text{per} K_n = j_n.$$  

Proof: We use mathematical induction on $n$, the size of $B_n$.

(1) The case $n = 2$.

$$\det B_2 = \det \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = 5 = j_2.$$  

(2) The case $k = k + 1$ with $k > 1$. Assuming $\det B_k = \text{per} K_k = j_k$ holds, we want to prove $\det B_{k+1} = \text{per} K_{k+1} = j_{k+1}$.

Because $\det B_{k+1} = \det T_{k+1} + 4 \det T_k - 4 \det T_{k-1}$

$$= \det T_k + 4 \det T_{k-1} + 4 \det T_{k-2} + 4 \det T_{k-3}.$$  

So if $n = k + 1$, the theorem holds. This completes the proof.
REFERENCES
