CHARACTERIZATIONS OF FUZZY CLOSURE SYSTEMS ON L-ORDERED SETS

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In this paper, we present equivalent characterizations of L-closure systems on lower bounded complete L-ordered sets and complete L-lattices, respectively. These results demonstrate the feasibility of notion of fuzzy closure system developed in our previous work (L.-K. Guo, G.-Q. Zhang, Q.-G. Li: Fuzzy closure systems on L-ordered sets. Math. Log. Quart. 57 (3) (2011), 281-291.). Also we develop the notions of L-closure systems closed for directed sups and L-closure operators preserving directed sups on L-ordered sets. The one-to-one correspondence between these two structures is derived when the underlying L-ordered set is directed complete.

KEYWORDS: Fuzzy closure system, L-Ordered set, Fuzzy directed subset.

1. INTRODUCTION

Closure operators and closure systems are important tools in many branches of mathematics such as topology, algebra, logic and order theory. In particular, the relationship among closure operators, closure systems and Galois connections have been established on ordinary partially ordered sets (for short, posets). Moreover, it has been shown that there exists a mutual correspondence between closure operators preserving directed sups and closure systems closed for directed sups when the underlying posets are directed complete [12].

There also has been a lot of work on discussing closure operators, closure systems in the fuzzy setting. In [1, 3, 9, 10, 11], fuzzy closure operators, fuzzy closure systems and fuzzy Galois connections have been proposed and investigated on fuzzy powersets, which can be viewed as the fuzzification of the corresponding notions on classical powersets. On the other hand, the notion of L-ordered set (or fuzzy poset) has been developed for different purposes [5, 6, 7, 8, 18, 19, 20, 21] and much effort has been made to study closure operators, closure systems and Galois connections on L-ordered sets. In particular, the notion of fuzzy closure system on complete L-lattices was introduced and studied in [16]. The concept of L-kernel system was defined on upper bounded complete L-ordered sets and the one-to-one correspondence between L-kernel systems and L-kernel operators was established in [23]. In addition, the notions of fuzzy closure operator and fuzzy Galois connection were developed on L-ordered sets and the connection between them was also studied in [2, 22]. However, these notions of fuzzy closure system and closure operator are all developed on special L-ordered sets. In contrast to earlier papers on this topic, we proposed the notion of fuzzy closure system on general L-ordered sets and investigated its connections with fuzzy closure operators and fuzzy Galois connections in [13].

This paper is an extension of our previous work. We first give equivalent characterizations of fuzzy closure systems on lower bounded complete L-ordered set and complete L-lattice respectively. These characterizations indicate that our notion of fuzzy closure system provides a more general framework in the sense that fuzzy closure systems developed in [16] and [23] can be viewed as special cases of that proposed in [13]. We moreover adopt fuzzy directed subsets in our discussion and propose the notions of L-closure systems closed for directed sups and L-closure operators preserving directed sups on...
L-ordered sets. It is shown that there exists a mutual correspondence between these two structures when the underlying L-ordered set is directed complete. This can be viewed as a generalization of the corresponding classical results in the fuzzy setting.

The paper is organized as follows: Section 2 recalls the basic notations and properties of L-ordered sets. In Section 3, equivalent characterizations of L-closure systems on lower bounded complete L-ordered sets and complete L-lattices are presented. Section 4 discusses the relationship between L-closure systems closed for directed sups and L-closure operators preserving directed sups on directed complete L-ordered sets. Section 5 gives the conclusions.

2. Preliminary

In fuzzy logic, residuated lattices play the important role of structure of truth values. In this paper, we choose complete residuated lattices as the algebraic structures of truth degrees. A complete residuated lattice is a structure $L = (\mathbb{L}, \wedge, \vee, \cdot, 0, 1)$ where (i) $(\mathbb{L}, \wedge, \vee, 0, 1)$ is a complete lattice, 0 is the least element and 1 is the greatest element; (ii) $(\mathbb{L}, \cdot, 1)$ is a commutative monoid, i.e., $\cdot: \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ is a commutative associative operator on $\mathbb{L}$, and $x \cdot 1 = x$ for any $x \in \mathbb{L}$; (iii) $\cdot: \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ is an operator on $\mathbb{L}$ and the equivalence $x \leq y \Rightarrow z \Leftrightarrow x \cdot y \leq z$ holds for any $x, y, z \in \mathbb{L}$. When $L = \{0, 1\}$, a complete residuated lattice, denoted as 2, can be given as: $a \wedge b = 1$ if and only if $a = b = 1$; $a \vee b = 0$ if and only if $a = b = 0$; $a \cdot b = 1$ if and only if $a = b = 1$; $a \rightarrow b = 0$ if and only if $a = 1$ and $b = 0$. More properties of complete residuated lattices can be found in [4].

We first recall some basic notations about L-sets. Suppose $X$ is a non-empty set and denote the set of all L-sets on $X$ by $L^X$, i.e., $L^X = \{ A | A : X \rightarrow L \}$. We call $L^X$ the L-powerset of $X$. For any $A, B \in L^X$, the subsethood degree of $A$ in $B$ is defined as $S(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$. For any $a \in \mathbb{L}$ and $A \in L^X$, L-sets $a \rightarrow A$ and $a \cdot A$ are defined as for any $x \in X$, $(a \rightarrow A)(x) = a \rightarrow A(x)$ and $(a \cdot A)(x) = a \cdot A(x)$, respectively. For any family $\{A_i | i \in I\} \subseteq L^X$, L-sets $\bigcup_{i \in I} A_i$ and $\bigcap_{i \in I} A_i$ are defined as for any $x \in X$, $\bigcup_{i \in I} A_i(x) = \bigvee_{i \in I} A_i(x)$ and $\bigcap_{i \in I} A_i(x) = \bigwedge_{i \in I} A_i(x)$, respectively.

**Definition 2.1:** [5, 8, 24] Let $X$ be a non-empty set. An L-order (or fuzzy order) on $X$ is a binary L-relation $e : X \times X \rightarrow L$ which satisfies

1. $e(x, x) = 1$ for any $x \in X$;
2. $e(x, y) e(y, z) = e(x, z)$ for any $x, y, z \in X$;
3. $e(x, y) = e(y, x)$ entails $x = y$ for any $x, y \in X$.

The pair $(X, e)$ is called an L-ordered set (or fuzzy poset).

**Example 2.1:** (1) Obviously, $(L^X, S)$ is an L-ordered set.

(2) Given an L-ordered set $(X, e)$, define $\leq \subseteq X \times X$ as $x \leq y \Leftrightarrow e(x, y) = 1$ for any $x, y \in X$. It is trivial to check that $\leq$ is an ordinary partial order on $X$. Conversely, for any ordinary partially ordered set $(X, \leq)$, define $e_{\leq} : X \times X \rightarrow L$ as $e_{\leq}(x, x') = 1$ if $x \leq x'$, and $e_{\leq}(x, x') = 0$ otherwise. It is easy to see that $e_{\leq}$ is an L-order on $X$. Moreover, when the truth value structure is chosen as $L = 2$, we have $e_{\leq} = e$ and $\leq = \leq$, which means that the classical partially ordered sets can be viewed as special L-ordered sets with $L = 2$.

(3) Suppose $(X, e)$ is an L-ordered set and $M$ is a non-empty subset of $X$. Then $(M, e)$ is also an L-ordered set, where the L-order $e$ on $M$ inherits from $(X, e)$.

We turn to recall some basic notations and results related to special L-ordered sets, namely lower bounded complete L-ordered sets and complete L-lattices. Let $A$ be an L-set on $X$, $x' \in X$ is called an lower bound of $A$ if $A(x) \leq e(x', x)$ for any $x \in X$. In this case, $A$ is said to be lower bounded. If there exists $x' \in X$
such that $A(x) = e(x', x)$ for any $x \in X$, then $A$ is called an *-singleton on $X$. Given an $L$-ordered set $(X, e)$, a pair of operators $(\vee, \wedge)$ on $L^X$ are defined as for any $A \in L^X$,

$$A^\vee(x) = \bigwedge_{y \in X} (A(y) \rightarrow e(y, x)),$$
$$A^\wedge(x) = \bigwedge_{y \in X} (A(y) \rightarrow e(y, x)).$$

For any $A \in L^X$, define $A^{\inf} : A \rightarrow L$ by $A^{\inf}(x) = A^\vee(x) \wedge A^\sup(x)$ for any $x \in X$; define $A^{\sup} : A \rightarrow L$ by $A^{\sup}(x) = A^\vee(x) \wedge A^{\sup}(x)$ for any $x \in X$.

**Theorem 2.1:** [5, 24] Let $(X, e)$ be an $L$-ordered set and $A \in L^X$. Then the following are equivalent:

1. $A^{\sup}$ (respectively, $A^{\inf}$) is an e-singleton.
2. There exists $x' \in X$ such that $A^{\sup}(x') = 1$ (respectively, $A^{\inf}(x') = 1$).
3. There exists $x' \in X$ such that $\bigwedge_{x \in X} (A(x) \rightarrow e(x, y)) = e(x', y)$ (respectively, $\bigwedge_{x \in X} (A(x) \rightarrow e(y, x)) = e(y, x')$) for any $y \in X$.
4. There exists $x' \in X$ such that $A(x) \leq e(x, x')$ (respectively, $A(x) \leq e(x', x)$) for any $x \in X$ and $\bigwedge_{x \in X} (A(x) \rightarrow e(x, y)) \leq e(x', y)$ (respectively, $\bigwedge_{x \in X} (A(x) \rightarrow e(y, x)) \leq e(y, x')$) for any $y \in X$.

Given an $L$-ordered set $(X, e)$, for any $A \in L^X$, if $x'$ described in Theorem 2.1 exists, then it is unique. We call it the join (respectively, meet) of $A$ in $(X, e)$ and denote it as $\sqcup A$ (respectively, $\sqcap A$).

**Definition 2.2:** [23] An $L$-ordered set is said to be lower bounded complete if its every lower bounded $L$-set has the meet.

**Definition 2.3:** [5, 20] Let $(X, e)$ be an $L$-ordered set. $(X, e)$ is called a complete $L$-lattice if both $\sqcup A$ and $\sqcap A$ exist for any $A \in L^X$.

**Example 2.2:** $(L^X, S)$ is a complete $L$-lattice. Indeed, suppose $A \in L^X$, for any $B \in L^X$, we have

$$S(\bigcup_{A \in L^X} (A(A) * A), B) = \bigwedge_{x \in X} \bigwedge_{A \in L^X} (A(A) * A(x)) \rightarrow B(x)$$
$$= \bigwedge_{x \in X} A(A) * A(x) \rightarrow B(x)$$
$$= \bigwedge_{A \in L^X} \bigwedge_{x \in X} (A(A) \rightarrow A(x) \rightarrow B(x))$$
$$= \bigwedge_{A \in L^X} A(A) \rightarrow \bigwedge_{x \in X} (A(x) \rightarrow B(x))$$
$$= \bigwedge_{A \in L^X} A(A) \rightarrow S(A, B).$$

By Theorem 2.1, $\bigcup_{A \in L^X} (A(A) * A)$ is the join of $A$ in $(L^X, S)$, i.e., $\sqcup A = \bigcup_{A \in L^X} (A(A) * A)$.

Furthermore,

$$S(B, \bigcap_{A \in L^X} (A(A) * A)) = \bigwedge_{x \in X} [B(x) \rightarrow \bigwedge_{A \in L^X} (A(A) \rightarrow A(x))]$$
$$= \bigwedge_{x \in X} B(x) \rightarrow \bigwedge_{A \in L^X} (A(A) \rightarrow A(x)).$$
To begin with, we first recall some basic notions and results of fuzzy closure systems on the general \( L \)-ordered sets and complete \( L \)-lattices which will be used as important tools in the subsequent section.

**Proposition 2.1:** [16] Let \((X, e)\) be a complete \( L \)-lattice. Then for any \( x \in X \) and \( \{x_t\}_{t \in T} \subseteq X \),

\[
e(x, \bigwedge_{t \in T} x_t) = \bigwedge_{t \in T} e(x, x_t), \quad e(\bigvee_{t \in T} x_t, x) = \bigvee_{t \in T} e(x_t, x),
\]

where \( \bigwedge_{t \in T} x_t \) and \( \bigvee_{t \in T} x_t \) are the meet and join of \( \{x_t\}_{t \in T} \) in \((X, \le)\), respectively.

**Theorem 2.2:** [15, 17] Let \((X, e)\) be an \( L \)-ordered set. Then the following are equivalent:

1. \((X, e)\) is a complete \( L \)-lattice.
2. \( \bigwedge A \) exists for any \( A \in L^X \).
3. \( \bigvee A \) exists for any \( A \in L^X \).

Finally, we recall the following two basic results of complete \( L \)-lattices which will be used as important tools in the subsequent section.

**Theorem 2.3:** [14, 17] An \( L \)-ordered set \((X, e)\) is a complete \( L \)-lattice if and only if

1. \( X \) is tensored in the sense that for any \( a \in L \) and \( x \in X \), there is an element \( a \otimes x \in X \), called the tensor of \( a \) with \( x \), such that for any \( y \in X \), \( e(a \otimes x, y) = a \rightarrow e(x, y) \);
2. \( X \) is cotensored in the sense that for any \( a \in L \) and \( x \in X \), there is an element \( a \rightarrow x \in X \), called the cotensor of \( a \) with \( x \), such that for any \( y \in X \), \( e(y, a \rightarrow x) = a \rightarrow e(y, x) \);
3. \((X, \le)\) is a complete lattice.

It is easy to verify that the tensor and contensor of \( a \) with \( x \) given in Theorem 2.3 are unique.

### 3. L-Closure Systems on Lower Bounded Complete \( L \)-Ordered Sets and Complete \( L \)-Lattices

In this section, we give equivalent characterizations of fuzzy closure systems on lower bounded complete \( L \)-ordered sets and complete \( L \)-lattices respectively. To begin with, we first recall some basic notions and results of fuzzy closure systems on the general \( L \)-ordered sets. The following definition is a slight simplification of that provided in [13].

**Definition 3.1:** Let \((X, e)\) be an \( L \)-ordered set and \( M \) a non-empty subset of \( X \). \( M \) is called an \( L \)-closure system on \( X \) if for any \( x \in X \), there exists \( m \in M \) such that \( e(x, m) = e(m, m) \) for any \( m \in M \). In this case, \( m \) is called the least upper bound of \( x \) in \( M \).

**Remark 3.1:** (1) It is trivial to check that for any \( L \)-closure system on \( X \), the least upper bound of \( x \in X \) is unique.
(2) It can be checked that for any \( L \)-closure system \( M \) on \( X \), \( m \) is the least upper bound of \( x \) in \( M \) if and only if \( e (x, m) = 1 \) and \( e (x, m) \leq e (m, m) \) for any \( m \in M \).

(3) Suppose the truth value structure is chosen as \( L = 2 \). It is easy to see that an \( L \)-closure system on \( X \) is exactly a classical closure system on \( (X, \leq) \). In this sense, the notion of \( L \)-closure system is really the generalization of the corresponding classical notion in the fuzzy setting.

**Example 3.1:** Let \( X \) be a non-empty set, and \( A = (A_i)_{i \in I} \) a subset of \( L^X \), where \( I \) is an index set. Introduced initially in [1], \( A = \{ A_i \in L^X \mid i \in I \} \) is said to be closed under \( S \)-intersections if \( \bigcap_{i \in I} (S (A_i) \rightarrow A_i) \in \wp \) for any \( A_i \in L^X \).

We have proved in [13] that \( \wp \) is an \( L \)-closure system in the sense of Definition 3.1 if and only if \( \wp \) is closed under \( S \)-intersections. Therefore, every system closed under \( S \)-intersections can be viewed as a special \( L \)-closure system.

**Definition 3.2:** [2] Let \((X, e_x)\) and \((Y, e_y)\) be \( L \)-ordered sets. A mapping \( f: X \rightarrow Y \) is said to be \( L \)-order preserving if for any \( x, x' \in X \),

\[
e_y (x, x') \leq e_y (f (x), f (x')).
\]

If a self-mapping \( f \) on a given \( L \)-ordered set \((X, e)\) is \( L \)-order preserving and satisfies \( f f = f \), then \( f \) is called an \( L \)-projection on \( X \). For convenience, we fix some notations as follows: for each mapping \( f: X \rightarrow Y \), a mapping \( f^*: L^X \rightarrow L^Y \), called the \( L \)-valued Zadeh function of \( f \), is constructed by \( f^*(A) (y) = \bigvee_x f (x) \) for any \( A \in L^X \) and \( y \in Y \). For any non-empty subset \( M \subseteq X \), the notation \( \overline{id}_M \) is used to denote the restriction of \( \overline{id}_X \) on \( M \). This means \( \overline{id}_M \) is a mapping from \( M \) to \( X \) which assigns to each \( m \in M \) itself. For any mapping \( f \) on \( X \), we use \( f \) to denote the co-restriction of \( f \) to the image \( f (X) \). That is, \( f \) is a mapping from \( X \) to \( f (X) \) which assigns to each \( x \in X \) the image \( f (x) \) of \( x \).

**Proposition 3.1:** Let \((X, e)\) be an \( L \)-ordered set, \( p: X \rightarrow X \) an \( L \)-projection on \( X \) and \( B \in L^X \). If \( \bigcup \overline{id}_{p (x)} (B) \) exists in \((X, e)\), then \( \bigcup B \) exists in \((p (X), e)\) and \( \bigcup B = p (\bigcup \overline{id}_{p (x)} (B)) \).

**Proof:** Let \( x' = \bigcup \overline{id}_{p (x)} (B) \). By Theorem 2.1, we only need to show that for any \( y \in p (X) \), \( e (p (x'), y) = \bigwedge \overline{(z \in p (X))} B (z) \rightarrow e (z, y) \), i.e., for any \( x \in X \), \( e (p (x'), p (x)) = \bigwedge \overline{(z \in X)} (B (p (z)) \rightarrow e (p (z), p (x))) \). Now fix some \( x \in X \).

On the one hand,

\[
\bigwedge \overline{(z \in X)} (B (p (z)) \rightarrow e (p (z), p (x))) = \bigwedge \overline{(z \in p (X))} (B (z) \rightarrow e (z, p (x)))
= \bigwedge \overline{(z \in X)} \overline{(z \in X)} (B (z) \rightarrow e (z, p (x)))
= e (x', p (x))
\leq e (p (x'), p (x)).
\]

On the other hand, suppose \( z \in X \). By Theorem 2.1, we have \( \overline{id}_{p (x)} (B) (p (z)) \rightarrow e (p (z), x') \), i.e., \( B (p (z)) \leq e (p (z), x') \). Since \( p \) is an \( L \)-projection, \( B (p (z)) \leq e (p (z), p (x')) \). It follows that \( e (p (x'), p (x)) B (p (z)) \leq e (p (x'), p (x)) \leq e (p (z), p (x')) \), which implies \( e (p (x'), p (x)) \leq B (p (z)) \rightarrow e (p (z), p (x)) \). We thus have \( e (p (x'), p (x)) \leq \bigwedge \overline{(z \in X)} (B (p (z)) \rightarrow e (p (z), p (x))) \).

**Definition 3.3:** [2] Let \((X, e)\) be an \( L \)-ordered set. An \( L \)-order preserving mapping \( f \) on \( X \) is called an \( L \)-closure operator if for any \( x \in X \),

\[
e (x, f (x)) = e (f f (x), f (x)) = 1.
\]

**Example 3.2:** Let \( X \) be a non-empty set. In [1, 3], the notion of \( L \)-closure operator on \( L^X \) was defined as a mapping \( C: L^X \rightarrow L^X \) where for any \( A \in L^X \), \( (1) S (A, C (A)) = 1 \); \( (2) S (A, A') \leq S (C (A), C (A')) \); \( (3) C (A) = C (C (A)) \). Apparently, \( C \) is an \( L \)-closure operator on \((L^X, S)\).
The following theorem show the one-to-one correspondence between $L$-closure operators and $L$-closure systems on $L$-ordered sets.

**Theorem 3.1:** [13] Let $(X, e)$ be an $L$-ordered set, $f$ an $L$-closure operator on $X$ and $M$ an $L$-closure system on $X$. Then $M = f(X)$ is an $L$-closure system on $X$ and $f_M$ is an $L$-closure operator on $X$, where $f_M$ is defined by associating each $x \in X$ with the least upper bound of $a \rightarrow x$ in $M$. Moreover, $f = f_M$ and $M = M_{f_M}$ i.e., mappings $f \mapsto f_M$ and $M \mapsto f_M$ are mutually inverse.

**Corollary 3.1:** Let $(X, e)$ be a complete $L$-lattice, $M$ an $L$-closure system $X$ and $f_M$ the corresponding $L$-closure operator of $M$. For any $x \in X$ and $m \in M$, denote the least upper bound of $a \rightarrow x$ in $M$ as $m_{a \rightarrow x}$ and the least upper bound of $a \otimes x$ in $M$ as $m_{a \otimes x}$. Then we have $m_{a \rightarrow x} = a \mapsto x$ is equivalent to $f_M(a \rightarrow x) = a \mapsto f_M(x)$; and $m_{a \otimes x} = a \otimes m_z$ is equivalent to $f_M(a \otimes x) = a \otimes f_M(x)$.

Now we turn to study $L$-closure systems on some special types of $L$-ordered sets. The following theorem provides an equivalent characterization of $L$-closure systems on lower bounded complete $L$-ordered sets.

**Theorem 3.2:** Let $(X, e)$ be lower bounded complete and $M$ a non-empty subset of $X$. Then $M$ is an $L$-closure system if and only if $\bigcap \tilde{id}_M(B) \in M$ for any $B \in L^M$ with $\bigcap \tilde{id}_M(B)$ exists in $(X, e)$.

**Proof:** Suppose $M$ is an $L$-closure system on $X$ and $B \in L^M$ with $\bigcap \tilde{id}_M(B)$ exists in $(X, e)$. Let $f_M$ be the corresponding $L$-closure operator of $M$ according to Theorem 3.1. For convenience, denote $x' = \bigcap \tilde{id}_M(B)$. Since $x' \in M$ is equivalent to $x' = f_M(x')$, we only need to verify that $e(x', f_M(x')) = e(f_M(x'), x') = 1$ to prove $x' \in M$. Apparently, $e(x', f_M(x')) = 1$ because $f_M$ is an $L$-closure operator. Furthermore, by Theorem 2.1, we have $e(f_M(x'), x') = \bigwedge_{x \in X} (\tilde{id}_M(B)(x) \rightarrow e(f_M(x'), x))$ which means $e(f_M(x'), x') = \bigwedge_{m \in M} (B(m) \rightarrow e(f_M(x'), m))$.

We only need to prove $B(m) \leq e(f_M(x'), m)$ for any $m \in M$, which is true: for any $m \in M$, by Theorem 2.1, $B(m) = \tilde{id}_M(B)(m) \leq e(x', m) \leq e(f_M(x'), m) = e(f_M(x'), m)$.

Conversely, suppose $x \in X$, define $B_x : M \rightarrow L$ as $B_x(m) = e(x, m)$ for any $m \in M$. It is trivial to check that $x$ is a lower bound of $\tilde{id}_M(B)$, which means $\tilde{id}_M(B)$ is lower bounded. Because $(X, e)$ is lower bounded complete, it follows that $\bigcap \tilde{id}_M(B_x) \in M$. By Theorem 2.1,

\[
e(x, \bigcap \tilde{id}_M(B_x)) = \bigwedge_{y < X} (\tilde{id}_M(B_x)(y) \rightarrow e(x, y))
= \bigwedge_{m \in M} (e(x, m) \rightarrow e(x, m))
= 1.
\]

Furthermore, for any $m \in M$, it follows from Theorem 2.1 that $e(x, m) = \tilde{id}_M(B_x)(m) \leq e(\bigcap \tilde{id}_M(B_x), m)$. Therefore, $\bigcap \tilde{id}_M(B_x)$ is exactly the least upper bound of $x$ in $M$ and thus $M$ is an $L$-closure system on $X$.

We turn to consider $L$-closure systems on complete $L$-lattices. In the following, we first review some basic notions and results of fuzzy Galois connections between $L$-ordered sets which play important role in studying closure systems and closure operators.

**Definition 3.4:** [22] Let $(X, e_X)$ and $(Y, e_Y)$ be two $L$-ordered sets, $\varphi : X \rightarrow$ and $\psi : Y \rightarrow X$ two $L$-order preserving mappings. The pair $(\varphi, \psi)$ is called an $L$-Galois connection if for any $x \in X$ and $y \in Y$,

\[
e_Y(\varphi(x), y) = e_X(x, (y)).
\]

For an $L$-Galois connection $(\varphi, \psi)$, $\varphi$ is called the left adjoint of $\psi$, and dually, $\psi$ the right adjoint of $\varphi$.

**Example 3.3:** A formal fuzzy context is a triplet $(X, Y, R)$, where $X, Y$ are nonempty sets and $R$ is a fuzzy relation from $X$ to $Y$, i.e., $R : X \times Y \rightarrow L$. In [11], a pair of operators $(R_{\geq}, R_{\leq})$ are defined as for any $A \in L^X$ and $B \in L^Y$, $A \geq_{R_{\geq}} B$ if $A \geq_{R_{\leq}} B$.
$R_\vdash : L^X \rightarrow L^Y$, $R_\vdash (A) (y) = \bigvee_{x \in X} (A \vdash x) \ast R(x, y)$,

$R^\vdash : L^Y \rightarrow L^X$, $R^\vdash (B)(x) = \bigwedge_{y \in Y} (R(x, y) \rightarrow B(y))$.

A pair $(P, Q) \in L^X \times L^Y$ is called a property oriented concept of $(X, Y, R)$ if $R_\vdash (P) = Q$ and $R^\vdash (Q) = P$. For any $A \in L^X$ and $B \in L^Y$,

$$S_\vdash (R_\vdash (A), B) = \bigwedge_{y \in Y} [R_\vdash (A)(y) \rightarrow B(y)]$$

This means $(R_\vdash, R^\vdash)$ is an $L$-Galois connection between $(L^X, S_\vdash)$ and $(L^Y, S_\vdash)$.

**Proposition 3.2:** [17] Let $(X, e)$ and $(Y, e)$ be complete $L$-lattices. Then

1. $f : X \rightarrow Y$ is a left adjoint of an $L$-Galois connection if and only if $f$ is a left adjoint from $(X, \leq_x)$ to $(Y, \leq_y)$ and preserves tensors in the sense that $f(a \otimes x) = a \otimes f(x)$;

2. $f : X \rightarrow Y$ is a right adjoint of an $L$-Galois connection if and only if $f$ is a right adjoint from $(X, \leq_x)$ to $(Y, \leq_y)$ and preserves cotensors in the sense that $f(a \rightarrow x) = a \rightarrow f(x)$.

The following theorem provides an equivalent characterization of $L$-closure systems on complete $L$-lattices.

**Theorem 3.3:** Let $(X, e)$ be a complete $L$-lattice, $M$ a non-empty subset of $X$. Then $M$ is an $L$-closure system on $X$ if and only if $\bigwedge_{i \in T} x_i \in M$ for any $\{x_i\}_{i \in T} \subseteq M$ and $a \rightarrow m \in M$ for any $a \in L$ and $m \in M$, where $\bigwedge_{i \in T} x_i$ is the meet of $\{x_i\}_{i \in T}$ in $(X, \leq)$.

**Proof:** Suppose $M$ is an $L$-closure system. Let $f_M$ be the corresponding $L$-closure operator of $M$ according to Theorem 3.1. Since $M = f_M(X)$ and $f_M$ is a $L$-projection, it is easy to see from Proposition 3.1 that $(M, e)$ is also a complete $L$-lattice. Moreover, since $e(m, t) = e(x, m)$ for any $x \in X$ and $m \in M$, it follows that $(\widehat{f_M}, \widehat{i_d}_M)$ forms an $L$-Galois connection between $(X, e)$ and $(M, e)$. Thus by Proposition 3.2, $\widehat{i_d}_M$ is a right adjoint from $(M, \leq)$ to $(X, \leq)$ and $\widehat{i_d}_M$ preserves cotensors. Since right adjoint preserves any meet, it holds that $\bigwedge_{i \in T} x_i = M$ for any $\{x_i\}_{i \in T} \subseteq M$. It is obvious that $a \rightarrow m \in M$ for any $m \in M$ and $a \in L$.

Conversely, suppose $B \in L^M$, let $m^* = \bigwedge_{m \in M} (B \vdash m) \Rightarrow m$ which is the meet of $\{B(m) \Rightarrow m \mid m \in M\}$ in $(X, \leq)$. By hypothesis, it is obvious that $m^* \in M$. In addition, for any $n \in M$,

$$e(n, m^*) = e(n, \bigwedge_{m \in M} (B \vdash m) \Rightarrow m))$$

$$= \bigwedge_{m \in M} e(n, B \vdash m) \Rightarrow m) \quad \text{(by Proposition 2.1)}$$

$$= \bigwedge_{m \in M} e(n, B \vdash m) \Rightarrow e(n, m)) \quad \text{(by Theorem 2.3)}.$$
Thus, by Theorem 2.1, \( m' \) is exactly the meet of \( B \) in \( (M, e) \), i.e., \( m' = \sqcap B \). It immediately follows from Theorem 2.2 that \( (M, e) \) is itself a complete \( L \)-lattice. For any \( \{ x_i \}_{i \in \tau} \subseteq M \), it is easy to see that \( \sqcap_{i \in \tau} x_i \) is exactly the meet of \( \{ x_i \}_{i \in \tau} \) in \( (M, \leq) \). This implies that \( \hat{i}_d_M \) is a right adjoint from \( (M, \leq) \) to \( (X, \leq) \). Moreover, it is easy to see that the cotensor of a with \( x \) in \( (M, e) \) is exactly the cotensor of \( a \) with \( x \) in \( (X, e) \). Thus, by Proposition 3.2, \( \hat{i}_d_M \) is a fuzzy right adjoint.

Let \( \varphi : X \longrightarrow M \) be the corresponding fuzzy left adjoint of \( \hat{i}_d_M \). Fix some \( x \in X \). We have \( e(x, \varphi(x)) = e(\varphi(x), \varphi(x)) = 1 \) by Definition 3.4. Moreover, for any \( m \in M \), by Definition 3.4 again, \( e(x, m) = e(x, \hat{i}_d_M(m)) = e(\varphi(x), m) \). Since \( e(m, \varphi(m)) = 1 \), we have \( e(x, m) = e(\varphi(x), m) \) \( e(m, \varphi(m)) \leq e(\varphi(x), \varphi(m)) \). Therefore, \( \varphi(x) \) is exactly the least upper bound of \( x \) in \( M \). Hence, \( M \) is an \( L \)-closure system on \( X \).

4. \( L \)-Closure Systems on \( L \)-dcpo's

Directed sets play important roles in Domain theory [12]. Much work has been done on the generalization of directed sets in the fuzzy setting [8, 15, 21, 24]. In this section, we adopt the notion of fuzzy directed subset in the investigation of fuzzy closure systems and give more characterizations of \( L \)-closure systems.

**Definition 4.1:** [21] Let \( (X, e) \) be an \( L \)-ordered set. \( D \in L^X \) is called a fuzzy directed subset of \( X \) if

1. \( \bigvee_{x \in X} D(x) = 1 \),
2. \( D(x) \ast D(x') \leq \bigvee_{z \in X} D(z) \ast e(x, z) \ast e(x', z) \) for any \( x, x' \in X \).

**Proposition 4.1:** [21] Let \( f \) be an \( L \)-order preserving mapping from \( (X, e_X) \) to \( (Y, e_Y) \) and \( D \) a fuzzy directed subset of \( X \). Then \( f^{-1}(D) \) is a fuzzy directed subset of \( Y \).

**Definition 4.2:** [21] An \( L \)-ordered set \( (X, e) \) is called a directed complete \( L \)-ordered set (shortly, \( L \)-dcpo) if \( \sqcup D \) exists for every fuzzy directed subset \( D \) of \( X \).

**Definition 4.3:** Let \( (X, e) \) be an \( L \)-ordered set and \( M \) a non-empty subset of \( X \). \( M \) is said to be closed for directed sups in \( X \) if for any fuzzy directed subset \( E \) of \( M \) with \( \sqcup \hat{i}_d_M(E) \) exists in \( (X, e) \), we have \( \sqcup E \) exists in \( (M, e) \) and \( \sqcup E = \sqcup \hat{i}_d_M(E) \).

**Definition 4.4:** Let \( f \) be an \( L \)-order preserving mapping from \( (X, e_X) \) to \( (Y, e_Y) \). \( f \) is said to preserve directed sups if for any fuzzy directed subset \( D \) of \( X \) with \( \sqcup D \) exists in \( (X, e_X) \), we have \( \sqcup f^{-1}(D) \) exists in \( (Y, e_Y) \) and

\[ f(\sqcup D) = \sqcup f^{-1}(D). \]

**Proposition 4.2:** Let \( (X, e) \) be an \( L \)-dcpo and \( p : X \longrightarrow X \) an \( L \)-projection on \( X \). Then \( (p(X), e) \) is also an \( L \)-dcpo. Moreover, if \( p \) preserves directed sups, then \( p(X) \) is closed for directed sups in \( X \).

**Proof:** \( (p(X), e) \) is an \( L \)-dcpo immediately follows from Proposition 3.1 and Proposition 4.1.

For the second part, suppose \( E \in L^p(X) \) is a fuzzy directed subset. Obviously, \( \hat{i}_d_{p(X)} \) is \( L \)-order preserving. Thus, by Proposition 4.1, \( \hat{i}_d_{p(X)}(E) \) is a fuzzy directed subset of \( X \). Since \( (X, e) \) is an \( L \)-dcpo, \( \sqcup \hat{i}_d_{p(X)}(E) \) exists in \( (X, e) \). Because \( p \) preserves directed sups, we have \( \sqcup p^{-1}(\hat{i}_d_{p(X)}(E)) \) exists in \( (X, e) \) and \( \sqcup p^{-1}(\hat{i}_d_{p(X)}(E)) = p(\sqcup \hat{i}_d_{p(X)}(E)) \). Furthermore, it is trivial to check that \( p^{-1}(\hat{i}_d_{p(X)}(E)) \) is a fuzzy directed subset of \( X \). If \( M \) is closed for directed sups in \( X \), then the corresponding \( L \)-closure operator \( f_M \), according to Theorem 3.1 preserves directed sups.
**Proof:** Suppose $D$ is a fuzzy directed subset of $X$. Because $(X, e)$ is an $L$-dcpo, $\square D$ exists in $(X, e)$. We first show that $f_M(\square D)$ is exactly the join of $\hat{f}_M(D)$ in $(M, e)$. Indeed, for any $y \in M$,

\[
\bigwedge_{z \in M} \left[ f_M^{-1}(D)(z) \rightarrow e(z, y) \right] = \bigwedge_{z \in M} \left( \bigvee_{f_M(x) = z} D(x) \rightarrow e(z, y) \right)
\]

\[
= \bigwedge_{z \in M} \bigwedge_{f_M(x) = z} \left[ D(x) \rightarrow e(z, y) \right]
\]

\[
= \bigwedge_{z \in M} \bigwedge_{f_M(x) = z} \left[ D(x) \rightarrow e(f_M(x), y) \right]
\]

\[
= \bigwedge_{x \in X} \left[ D(x) \rightarrow e(f_M(x), y) \right]
\]

\[
= \bigwedge_{x \in X} \left[ D(x) \rightarrow e(x, y) \right]
\]

\[
= e(\square D, y)
\]

\[
= e(f_M(\square D), y).
\]

By Theorem 2.1, we have $\square \hat{f}_M(D)$ exists and $\square \hat{f}_M(D) = f_M(\square D)$.

Furthermore, it is easy to verify that $f_M^{-1}(D) = \hat{D}$. As $(X, e)$ is an $L$-dcpo, we have $\square f_M^{-1}(D)$ exists and $\square f_M^{-1}(D) = \bigcup \hat{i}D = \square \hat{D}$. Since $M$ is closed for directed sups in $X$, we have $\square \hat{D}$ exists in $(M, e)$ and $\square f_M^{-1}(D) = \square \hat{D}$, which follows that $\square \hat{D} = \square \hat{f}_M^{-1}(D)$. Therefore, we obtain $f_M^{-1}(D) = f_M(\square D)$, which means that $f_M$ preserves directed sups.

**Proposition 4.4:** Let $(X, e)$ be an $L$-ordered set and $f$ an $L$-closure operator on $X$. If $f$ preserves directed sups, then $f(X)$ is an $L$-closure system which is closed for directed sups in $X$.

**Proof:** It immediately follows from Theorem 3.1 and Proposition 4.2.

**Theorem 4.1:** Let $(X, e)$ be an $L$-dcpo. Then the correspondence between $L$-closure operators and $L$-closure systems given in Theorem 3.1 holds for the set of $L$-closure operators preserving directed sups and the set of $L$-closure systems which are closed for directed sups.

**Proof:** It follows from Proposition 4.3 and Proposition 4.4.

**Corollary 4.1:** Let $(X, e)$ be an $L$-dcpo. An $L$-closure system $M$ on $X$ is closed for directed sups if and only if there exists an $L$-closure operator $f$ on $X$ such that $f$ preserves directed sups and $f(X) = M$.

**5. Conclusions**

This paper provided a simplification of the notion of fuzzy closure system on $L$-ordered sets which was developed in our previous work. It may be viewed as the most general extension of closure systems in the sense that the existing notions of fuzzy closure system in the literature can be viewed as special cases. We further gave equivalent characterizations of fuzzy closure systems on lower bounded complete $L$-ordered set and complete $L$-lattice, respectively. These characterizations indicate that our notion of fuzzy closure system provides a more feasible framework in studying closure systems. Finally, we employed fuzzy directed subsets in our discussion and proposed the notions of $L$-closure systems closed for directed sups and $L$-closure operators preserving directed sups on $L$-ordered sets.

The fact of mutual correspondence between these two structures when the underlying $L$-ordered set is directed complete can be viewed as a generalization of the corresponding classical results.
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